Non-chainable continua and Lelek's problem

by

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Abstract

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The set of compact connected metric spaces (continua) can be divided into classes according to the complexity of their descriptions as inverse limits of polyhedra. The simplest such class is the collection of chainable continua, i.e. those which are inverse limits of arcs.

In 1964, A. Lelek introduced a notion which is related to chainability, called span zero. A continuum X has span zero if any two continuous maps from any other continuum to X with identical ranges have a coincidence point. Lelek observed that every chainable continuum has span zero; he later asked whether span zero is in fact a characterization of chainability.

In this thesis, we construct a non-chainable continuum in the plane which has span zero, thus providing a counterexample for what is now known as Lelek's Problem in continuum theory. Moreover, we show that the plane contains an uncountable family of pairwise disjoint copies of this continuum. We discuss connections with the classical problem of determining up to homeomorphism all the homogeneous continua in the plane.

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Contents

Chapter 1

Introduction

The results of this thesis concern the theory of continua. Here, a *continuum* is a compact connected metric space, except in Chapter 2 where we consider the more general class of *Hausdorff continua*, where "metric" is weakened to "Hausdorff". If X is a metric continuum, we will invariably refer to the metric by d.

A standard way to produce examples in continuum theory is by taking inverse limits. Given spaces X_i and maps $f_i : X_{i+1} \to X_i$ ($i \in \omega$), the corresponding *inverse limit* is the space $\varprojlim\{X_i, f_i\} = \{\langle x_i \rangle_{i\in\omega} \in \prod_{i\in\omega} X_i : f_i(x_{i+1}) = x_i \text{ for each } i \in \omega\}$. The spaces X_i are called the *factor spaces*, and the maps f_i are the *bonding maps* of the inverse limit.

It is known (see [35]) that every continuum is an inverse limit of compact connected polyhedra (with onto bonding maps). This suggests a notion of complexity for continua according to how complex the factor spaces in its inverse limit description are. By this measure, the simplest class of continua are those which are inverse limits of *arcs*, i.e. inverse limits in which each factor space is homeomorphic to the interval $[0, 1] \subset \mathbb{R}$. Such continua are called *chainable* (also called *arc-like* or *snake-like*).

There are two other useful characterizations of chainability, one in terms of maps and one in terms of covers:

- A continuum X is chainable if and only if for every $\varepsilon > 0$, there is a map $f : X \to [0, 1]$ whose point-inverses have diameters $< \varepsilon$.

- A continuum X is chainable if and only if for every $\varepsilon > 0$, there is an open cover of X of mesh $< \varepsilon$ which forms a *chain*, i.e. a cover $\{U_1, \ldots U_n\}$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

The latter characterization is the source of the term "chainable".

Some important fundamental examples from continuum theory are chainable, including the $\sin(\frac{1}{x})$-continuum (see [35], Example 1.5), Knaster's buckethandle continuum (see [22], Example 1 of Section 48,V), and the pseudo-arc (see the discussion below).

The class of chainable continua is classical and well-studied in continuum theory. For this reason, much information on chainable continua is readily available. To illustrate this point, a (non-exhaustive) survey of some basic known properties of chainable continua is given below. References are provided for the deeper results; the others can be found e.g. in [35]. See also Bing's theorem below in the discussion of homogeneous continua for another fundamental fact about chainable continua.

If X is a chainable continuum, then the following statements hold for X:

(i) X can be embedded in the plane $\mathbb{R}^2$. More generally, any product of n chainable continua can be embedded in $\mathbb{R}^{n+1}$ [2].

(ii) Every subcontinuum of X is chainable.

(iii) X is atriodic, i.e. it contains no triods. A *triod* is a continuum T which has a subcontinuum Z such that $T \setminus Z$ is the union of three non-empty, mutually separated sets.

(iv) X is unicoherent, i.e. any two subcontinua whose union is X have connected intersection.

(v) X has the fixed point property, i.e. if $f : X \to X$ is continuous, then there is some point $p \in X$ with $f(p) = p$. In fact, any product of chainable continua has the fixed point property [10].

(vi) X is irreducible between two of its points, i.e. there are $p, q \in X$ such that no proper subcontinuum of X contains both p and q.

(vii) Every open image ([44]) and every monotone image ([4]) of X is chainable. A continuous map $f : X \to Y$ is *monotone* provided $f^{-1}(q)$ is connected for every $q \in Y$.

(viii) X is a continuous image of the pseudo-arc [29].

The main body of this thesis is concerned with still another property of chainable continua, which we discuss now. If X is a chainable continuum, then X has the following so-called universality property for maps:

(∗) If $f, g : C \to X$ are continuous maps from a continuum C to X with $f(C) = g(C)$, then there exists a point $c \in C$ with $f(c) = g(c)$.

J. F. Davis showed in [9] that this is equivalent to the same property with "$f(C) = g(C)$" replaced by "$f(C) \subseteq g(C)$". By taking $C = X$ and $g = \mathrm{id}_X$, it can be seen that this variant, and hence (∗), is a natural strengthening of the fixed point property.

In [23], A. Lelek initiated a study of those continua which satisfy the property (∗). Given a continuum X with metric d, he considered the following quantity:

$$\sup_{C,f,g} \inf_{c_1,c_2 \in C} d(f(c_1), g(c_2))$$

where the supremum is taken over all continua C and all continuous maps $f, g : C \to X$ with $f(C) = g(C)$. This number can be thought of as measuring how far the continuum X is from satisfying (∗). It is not difficult to see that in this supremum we may assume $C \subset X \times X$ and that f and g are the first and second coordinate projections, respectively, from $X \times X$ to X (given C, f, g, consider the image of C under the map $c \mapsto (f(c), g(c)) \in X \times X$). Hence the number given above is equal to the following, which Lelek called the *span of* X:

Definition.

$$\mathrm{Span}(X) = \sup_{Z} \inf_{(x_1,x_2) \in Z} d(x_1, x_2)$$

where the supremum is taken over all subcontinua $Z \subset X \times X$ with $\pi_1(Z) = \pi_2(Z)$ (here π_1 and π_2 are the first and second coordinate projections, respectively).

Lelek observed that chainable continua have span equal to zero, which is to say they satisfy the property (∗) given above. In [24], he asked whether this property is a characterization of chainability:

Question (Lelek, [24])**.** *Is it true that if a continuum X has span zero, then X is chainable?*

This is known as Lelek's problem, and has become a topic of much interest in continuum theory. It has been featured in a number of recent surveys, appearing as Problem 8 in [7], Problem 2 in [16], Problem 81 in [8], Conjecture 2 in [27], and in [35, p. 255]. Much of this attention is due to the fact that there are currently few available techniques to determine whether a given continuum X is chainable, and in some cases it is easier to check whether X has span zero.

One such instance occurs in the study of homogeneous continua in the plane. A topological space X is *homogeneous* if for every pair of points $x, y \in X$, there is a homeomorphism $h : X \to X$ such that $h(x) = y$. Mislove & Rogers proved in [30, 31] that any homogeneous compact metric space is homeomorphic to a product $X \times Y$, where X is a homogeneous continuum and Y is either a finite discrete space or the middle-thirds Cantor set. Thus, the study of homogeneous compacta effectively reduces to the study of homogeneous continua.

It has long been an open problem to classify all homogeneous continua in the plane $\mathbb{R}^2$ up to homeomorphism. This question was raised by Knaster & Kuratowski as Problème 2 in the first volume of Fundamenta Mathematicae [21] (for comparison, Problème 3 of this list is what is now known as Souslin's Hypothesis about linear orders). See [43] for a more recent survey of this and related problems.

At present there are exactly four known homogeneous continua in the plane:

(i) The **single point** (degenerate) continuum.

(ii) The **circle** $\mathbb{S}^1$.

(iii) The **pseudo-arc**. This was first discovered by Knaster in [20], where he constructed the first example of a non-degenerate hereditarily indecomposable continuum. A continuum is *indecomposable* if it is not the union of two of its proper subcontinua, and *hereditarily indecomposable* if each of its subcontinua is indecomposable.

The name "pseudo-arc" was given 26 years later by Moise to his example in [33] of a non-degenerate continuum other than the arc which is homeomorphic to each of its non-degenerate subcontinua. Moise's example was proved to be homogeneous by Bing in [3]. Bing later proved in [4] that the examples of Knaster and Moise are in fact homeomorphic (see Bing's theorem quoted below). He also observed that if S is either the Hilbert cube $[0, 1]^\omega$ or a Euclidean space $\mathbb{R}^n$ ($n \geq 2$), then most subcontinua of S are pseudo-arcs, in the sense that the set of continua homeomorphic to the pseudo-arc is a dense G_δ in the hyperspace of subcontinua of S.

(iv) The **circle of pseudo-arcs**. This was first described by Bing & Jones in [5]. Given a continuum X, by a *circle of X's* we mean a continuum Y which admits a continuous decomposition, each element of which is homeomorphic to X, for which the corresponding quotient of Y is homeomorphic to the circle $\mathbb{S}^1$. The circle of pseudo-arcs should not be confused with the *pseudo-circle*, also constructed by Bing [4], which has been proved to be not homogeneous by Rogers in [42].

Much effort has been put into determining whether this list contains all the homogeneous plane continua. Potential as yet undiscovered examples are typically classified according to whether they are decomposable or indecomposable, and whether they do or do not separate the plane. Rather than try to give a complete history of the work done on homogeneous continua in the plane, we instead summarize the most substantial and current results below.

Theorem (Jones, [18])**.** *If X is a decomposable homogeneous continuum in the plane $\mathbb{R}^2$, then X is a circle of mutually homeomorphic indecomposable homogeneous continua.*

In particular, Jones' theorem implies that if there is a homogeneous plane continuum which is not in the above list, then there is one which is indecomposable.

Theorem (Hagopian, [12])**.** *If X is an indecomposable homogeneous continuum in the plane $\mathbb{R}^2$, then X is hereditarily indecomposable.*

Theorem (Oversteegen & Tymchatyn, [37])**.** *If X is an indecomposable homogeneous continuum in the plane $\mathbb{R}^2$, then X has span zero.*

It can now be made explicit how Lelek's problem pertains to the problem of classifying all homogeneous plane continua. If "span zero" could be improved to "chainable" in this result of Oversteegen & Tymchatyn, then the classification would be complete due to the following theorem of Bing:

Theorem (Bing, [4])**.** *If X is a non-degenerate hereditarily indecomposable chainable continuum, then X is homeomorphic to the pseudo-arc.*

There is another approach to homogeneous continua in the plane which involves investigating their subcontinua. If $p \in X$, the *composant* of p in X is the set $\{q \in X :$ there is a proper subcontinuum $Y \subset X$ with $p, q \in Y\}$. It is well known that indecomposable continua have uncountably many composants, and these composants are pairwise disjoint. Hence if Y is a proper subcontinuum of a homogeneous indecomposable continuum, then X contains uncountably many pairwise disjoint copies of Y (one in each composant). Combined with the theorem of R. L. Moore [34] which states that the plane does not contain uncountably many pairwise disjoint triods, it follows that any homogeneous indecomposable continuum in the plane is atriodic.

One might try to mimic this approach to obtain that a homogeneous indecomposable continuum X in the plane contains no non-chainable subcontinua. This would mean every proper subcontinuum of X is chainable and hereditarily indecomposable (by Hagopian's theorem above), hence a pseudo-arc by Bing's theorem. By results of Oversteegen & Tymchatyn [37, 38, 39], it would follow that X is itself a pseudo-arc. However, in Chapter 4 we show that the plane does contain uncountably many copies of the non-chainable tree-like continuum from Chapter 3. Hence, a more sophisticated approach is required if one is to deduce that X contains no non-chainable subcontinua.

A number of properties of chainable continua have been established for span zero continua. We give a list of some of these for comparison with the above list for chainable continua. As before, references are given for the deeper results; the others are straightforward.

If X is a continuum with span zero, then the following statements hold for X:

(i) X is tree-like [39], i.e. X is an inverse limit of trees.

(ii) Every subcontinuum of X has span zero.

(iii) X is atriodic [23].

(iv) X is unicoherent [9].

(v) X has the fixed point property. Marsh [26] has proved that in fact any product of span zero continua has the fixed point property.

(vi) X is irreducible between two of its points [9].

(vii) Every open image ([19]) and every monotone image ([15]) of X has span zero.

(viii) X is a continuous image of the pseudo-arc [39].

Further, Bustamante et al. prove in [6] theorems about fixed point and universality properties in the hyperspace of subcontinua of a span zero continuum, generalizing corresponding theorems for chainable continua.

At present there is no analogue of Bing's theorem quoted above for continua of span zero – see Question 2 in Chapter 3.

There has been previous work toward finding a counterexample for Lelek's problem. Repovš et al. exhibit in [41] a sequence of trees in the plane with arbitrarily small (positive) spans, none of which has a chain cover of mesh < 1. In [1], Bartošová et al. consider generalizations of the notions of chainability and span zero to the class of Hausdorff continua, and prove via a model-theoretic construction that a counterexample for Lelek's problem in that context would imply that there exists a metric counterexample (see Chapter 2 of this thesis for an exposition of the application of model theory and a weaker variant of this result)

Many positive partial results for Lelek's problem have been obtained in [28], [36], [37], and [40]. Of particular note are the following two results of Minc [28] and Oversteegen [36], respectively:

- If X has span zero and X is an inverse limit of trees with simplicial bonding maps, then X is chainable. Given a sequence $\langle T_i \rangle_{i\in\omega}$ of trees, the maps $\langle f_i \rangle_{i\in\omega}$ are *simplicial* bonding maps provided there are fixed finite subsets $A_i \subset T_i$ such that if $a_1, a_2 \in A_{i+1}$ are adjacent in A_{i+1} (i.e. the arc $[a_1, a_2]$ in T_{i+1} joining a_1 and a_2 meets no other point of A_{i+1}), then $f_i(a_1)$ and $f_i(a_2)$ are adjacent in A_i, and f_i maps the arc $[a_1, a_2]$ linearly onto $[f_i(a_1), f_i(a_2)]$.

- If X has span zero and X is the image of the pseudo-arc under an induced map, then X is chainable. Given two inverse limit systems $\{X_i, f_i\}_{i\in\omega}$ and $\{Y_i, g_i\}_{i\in\omega}$, an *induced map* from $\varprojlim\{X_i, f_i\}$ to $\varprojlim\{Y_i, g_i\}$ is a function given coordinate-wise by a sequence of maps $\phi_i : X_i \to Y_i$ satisfying $g_i \circ \phi_{i+1} = \phi_i \circ f_i$ for each $i \in \omega$.

The former result explains the need for the infinitely many symbols of the form b_t in Chapter 3.

Chapters 2 and 3 of this thesis are devoted to the project of finding a counterexample for Lelek's problem. In Chapter 2 we describe an approach discussed by Van der Steeg in [47] which uses methods of model theory to construct a metric continuum from a given Hausdorff continuum. Van der Steeg proved that if the given Hausdorff continuum is non-chainable, then so is the resulting metric continuum; we will show that the same goes for a certain weaker version of span zero.

Chapter 3 constitutes the main body of this thesis. There we explicitly describe a counterexample for Lelek's problem in the plane. This involves some delicate placement of certain trees in the plane, for which we devise a scheme for combinatorial descriptions of embeddings of graphs using words. The main difficulty in the verification that the example described there is a counterexample for Lelek's problem comes down to checking that certain trees in the plane do not admit chain covers of small mesh. We reduce this to a combinatorial problem involving the word descriptions of these trees.

In Chapter 4 we prove that the example X from Chapter 3 has the property that $X \times \mathcal{C}$ embeds in the plane $\mathbb{R}^2$, where $\mathcal{C}$ is the middle-thirds Cantor set. This demonstrates that the result of Moore stating that the plane does not contain uncountably many pairwise disjoint triods [34] cannot be extended to copies of an arbitrary tree-like non-chainable continuum.

Chapter 2

Model-theoretic approach

2.1 Variants of span

Since the notion of span was introduced in [23], a number of variants of the original definition have been investigated. Some of these are collected in [25]; we reproduce the definitions here.

Definition. Let X be a metric continuum (with metric d), and let $\mathcal{P}(Z)$ be a property of a subcontinuum $Z \subset X \times X$. The number

$$\sup\{\inf_{(x_1,x_2)\in Z} d(x_1, x_2) : Z \subset X \times X \text{ is a subcontinuum such that } \mathcal{P}(Z) \text{ holds}\}$$

is called the

- *span of* X if $\mathcal{P}(Z)$ is $\pi_1(Z) = \pi_2(Z)$;
- *semispan of* X if $\mathcal{P}(Z)$ is $\pi_1(Z) \subseteq \pi_2(Z)$;
- *surjective span of* X if $\mathcal{P}(Z)$ is $\pi_1(Z) = \pi_2(Z) = X$;
- *surjective semispan of* X if $\mathcal{P}(Z)$ is $\pi_2(Z) = X$.

It is shown in [13] that no two of these versions of span are always equal, even among the class of circles or among the class of simple triods. However, Davis proves in [9] that X has span zero if and only if X has semispan zero. See also Question 1 below.

Observe that X has a certain type of span equal to zero if and only if every subcontinuum $Z \subset X \times X$ satisfying the corresponding property $\mathcal{P}(Z)$ meets the diagonal ΔX. This equivalence makes it clear that having any one of the spans equal to zero is preserved under homeomorphisms. It also affords a natural definition of (surjective) (semi)span zero for Hausdorff continua:

Definition. A Hausdorff continuum X has (surjective) (semi)span zero if every subcontinuum $Z \subset X \times X$ satisfying the corresponding property $\mathcal{P}(Z)$ (from the above definition) meets the diagonal ΔX.

We also generalize the definition of chainability to the class of Hausdorff continua:

Definition. A Hausdorff continuum X is chainable provided every open cover for X has an open refinement which forms a chain.

2.2 From compact Hausdorff to compact metric

By a *formula*, we will mean a first-order formula in the language of set theory (the first-order language whose only non-logical symbol is $\in$). If ϕ is a formula, we will write $\phi(x_1, \ldots, x_n)$ to indicate that the free variables in ϕ are among $x_1, \ldots, x_n$. Given a set H, we say H *models* ϕ *at* $a_1, \ldots, a_n \in H$, and write $H \models \phi[a_1, \ldots, a_n]$, provided ϕ holds with a_i substituted for each free occurance of x_i, and with all quantifiers in ϕ restricted to H.

An *elementary submodel* of H is a set $M \subseteq H$ with the property that for every formula $\phi(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in M$, we have $H \models \phi[a_1, \ldots, a_n]$ if and only if $M \models \phi[a_1, \ldots, a_n]$. The Löwenheim-Skolem Theorem of model theory (see [17], Theorem 12.1) affords, given any countable subset $S \subset H$, a countable elementary submodel M of H with $S \subseteq M$.

By a *lattice*, we mean a partially ordered set in which any two elements have a least upper bound and a greatest lower bound. A *base for the closed sets* of a topological space X is a family $\mathcal{A}$ of closed sets such that for any closed $K \subseteq X$, there is a subfamily $\mathcal{A}_K \subseteq \mathcal{A}$ with $K = \bigcap \mathcal{A}_K$.

We will make use of the following reformulation of a theorem of Wallman from [48], which is an analogue of the Stone Representation Theorem for Boolean algebras:

Theorem (Wallman's Representation Theorem)**.** *There is a formula $\alpha(x)$, with all quantifiers bounded by x, such that $\alpha[L]$ holds if and only if L is (isomorphic to) a lattice base for the closed sets of a compact Hausdorff space.*

If L is a lattice satisfying $\alpha[L]$, we denote by wL the compact Hausdorff space afforded by Wallman's Theorem. The space wL is called the *Wallman representation of* L, and can be defined as the set of ultrafilters on L, with topology defined by basic open sets of the form $U_A := \{u : u \text{ is an ultrafilter on } L \text{ with } A \notin u\}$, for $A \in L$. We will typically identify $A \in L$ with the basic closed set $wL \setminus U_A$.

The explicit definition of the formula α from Wallman's Theorem may be gleaned from [48] (see also [47]). For our purposes it is enough just to know that such a formula exists.

We are now ready to describe the construction from [47]. Suppose X is a compact Hausdorff space. Let H be some large set which contains all the objects we consider in the arguments to follow (for example we will need $X, 2^X, 2^{X\times X} \in H$ and $2^X, 2^{X\times X} \subset H$, and that all finite subsets of 2^X are in H, etc.). It is typical to take H to be the set of all sets of hereditary cardinality $< \theta$, for some sufficiently large cardinal θ.

Apply the Löwenheim-Skolem Theorem to obtain a countable elementary submodel M of H with $X, 2^X, 2^{X\times X} \in M$. Define $L = 2^X \cap M$ and $K = 2^{X\times X} \cap M$.

From Wallman's Representation Theorem, we obtain a compact Hausdorff space wL for which L is a base for the closed sets. Since $L \subset M$, we have that L is countable. Therefore, wL is metrizable by Urysohn's metrization theorem (see [11], Theorem 4.2.8).

A *shrinking* of an open cover $\langle U_i \rangle_{i\in\mathcal{I}}$ is an open cover $\langle V_i \rangle_{i\in\mathcal{I}}$ such that $V_i \subseteq U_i$ for each $i \in \mathcal{I}$. A standard fact about shrinkings of covers (Theorem 7.1.5 of [11]) easily implies the following:

Proposition. *Suppose Y is a compact Hausdorff space and J is a lattice base for the closed sets of Y. If $\mathcal{U}$ is an open cover for Y, then there is a shrinking of $\mathcal{U}$ by sets whose complements belong to J.*

One simple application of this fact is that every clopen subset of Y belongs to the lattice base J.

Lemma 1 (Theorem 2.14 of [47])**.** *If X is connected, then so is wL.*

Proof. X being connected means there is no non-trivial clopen subset of X, so H models the following formula:

$$\neg\,(\exists A \in 2^X)\,(\exists B \in 2^X)\,[A \neq \emptyset \ \wedge\ B \neq \emptyset \ \wedge\ A \cup B = X \ \wedge\ A \cap B = \emptyset]$$

By elementarity, M also models this formula, which means there is no pair of complementary non-empty closed sets $A, B \in L$. Thus there is no non-trivial clopen subset of wL which belongs to the lattice base L; by the above observation, this means there is no non-trivial clopen subset of wL. $\square$

Another application of the above Proposition is that the (non-)chainability of a space Y is determined within any lattice base J for the closed sets of Y. That is, Y is chainable if and only if given any $A_1, \ldots, A_n \in J$ whose complements cover Y, there are sets $B_1, \ldots, B_m \in J$ whose complements refine this cover and form a chain.

Lemma 2 (Lemma 3.7 & Section 7.3 of [47]). *If X is a non-chainable (Hausdorff) continuum, then wL is a non-chainable continuum.*

Proof. Suppose X is non-chainable. Then H models that there are $A_1, \ldots, A_n \in 2^X$ whose complements cover X and for whom there is no finite subset of 2^X of elements whose complements refine this cover and form a chain. By elementarity, there exist such $A_1, \ldots, A_n \in M$. That is, $wL \setminus A_1, \ldots, wL \setminus A_n$ covers wL, and there is no finite subset of L of elements whose complements refine this cover and form a chain. By the above remark, this implies wL is non-chainable. $\square$

Likewise one can prove that if wL is chainable, then so is wL (Section 7.2 of [47]).

We will also need the following lemma, whose proof we omit here:

Lemma 3 (Lemma 3.49 of [47]). *$wK \approx wL \times wL$, and this homeomorphism keeps sets of the form $A \times B$ invariant, for $A, B \in L$.*

2.3 Reflecting surjective (semi)span zero

Theorem 4. *Let X, M, L, K be as above. If X has surjective span zero, then so does wL.*

Proof. Let $Z \subset wL \times wL$ be a subcontinuum which misses the diagonal ΔwL. Find a set $Y \in K$ missing the diagonal of $wK \approx wL \times wL$ such that $Z \subseteq Y$.

Since X has surjective span zero, each component C of $Y \subset X \times X$ misses a line $L_C \subset X \times X$ of the form $X \times \{p_C\}$ or $\{p_C\} \times X$ ($p_C \in X$). Then we can find a set $H_C \in 2^{X \times X}$ such that $C \subseteq H_C \subseteq Y$, H_C is clopen in Y, and $H_C \cap L_C = \emptyset$. This yields a finite cover $H_{C_1}, \ldots, H_{C_n}$ of Y by sets in $2^{X \times X}$, none of which maps onto X under both projections; that is to say for each i there is a non-empty $P_i \in 2^X$ such that either $H_{C_i} \cap (X \times P_i)$ or $H_{C_i} \cap (P_i \times X)$ is empty.

By elementarity, there exist such elements H_{C_i} in K and P_i in L. Since Z must be contained in one element of this partition, we see that Z does not map onto wL under both projections. □

The above proof can be easily adapted to show that if X has surjective semispan zero, then so does wL (consider only lines of the form $L_C = X \times \{p_C\}$). However, it is not evident how to develop a similar argument to handle the case of span zero (or semispan zero).

In [1], Bartošová et al. succeed in reflecting span zero to a metric continuum by using a slightly more complex construction, and invoking more powerful machinery, namely Shelah's Ultrapower Isomorphism Theorem from [45]. The disparity in difficulty between these two reflection results raises the following question of Lelek:

Question 1 (Lelek, Problem 59 of [8])**.** *Does X have span zero (respectively, semispan zero) if and only if X has surjective span zero (respectively, surjective semispan zero)?*

Theorem 4 above and the other reflection results obtained in [47] imply that if there is a counterexample for Question 1 among all Hausdorff continua, then there is a metric one.

In connection with Question 1, Lelek has asked (Problems 1 & 2 of [25]) whether the (semi)span of X is bounded by twice the surjective (semi)span. Figure 2.1 is a computer generated image of a metric on the noose space (a circle with an arc attached by one endpoint) for which the span and semispan are each equal to 1, while the surjective span and surjective semispan are $\frac{1}{4}$.

It is still unknown whether there is any k (evidently ≥ 4) for which the (semi)span of X is bounded by k times the surjective (semi)span of X, for all continua X. If such a k exists, this would yield an affirmative answer to Question 1.

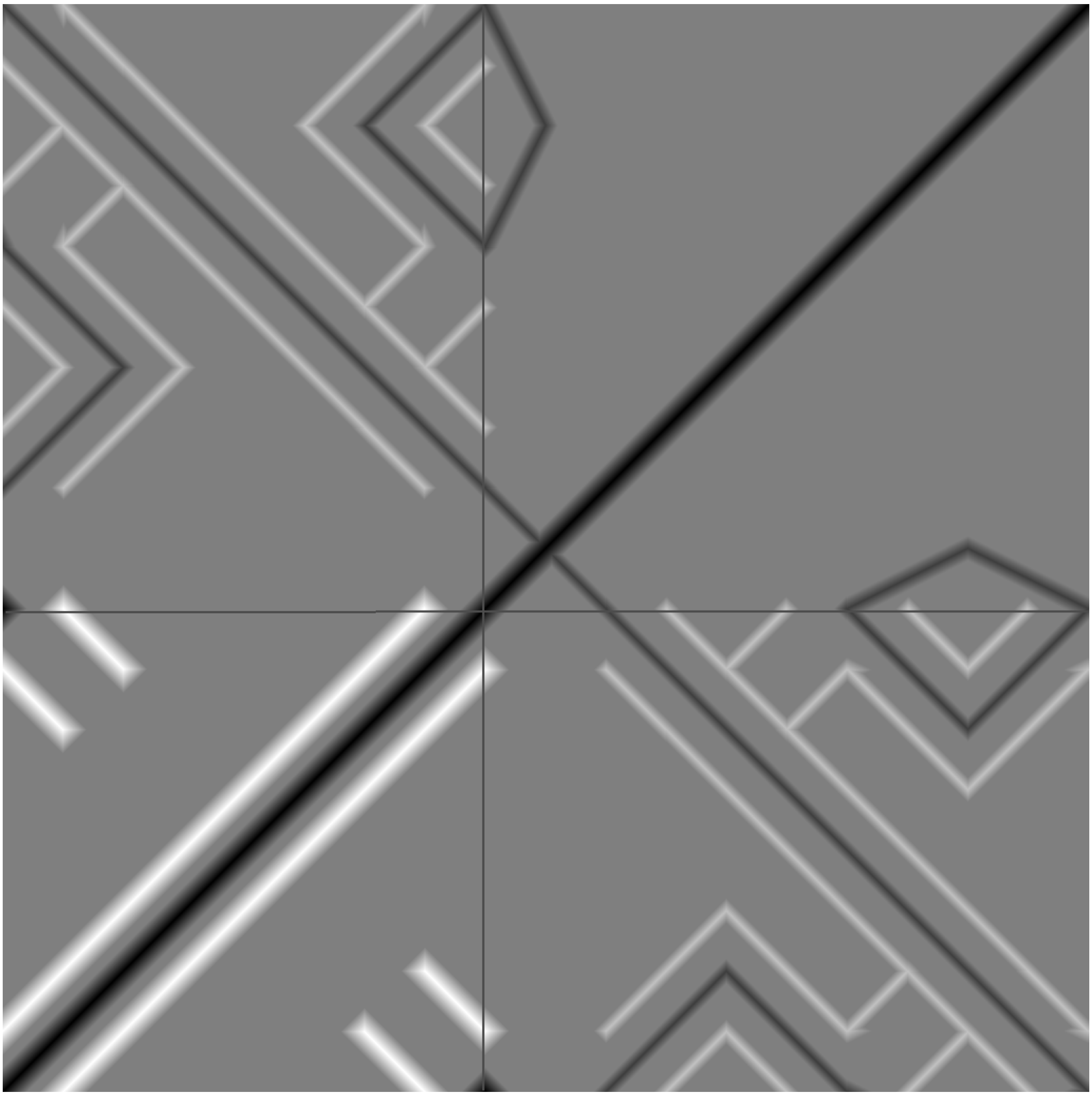


Figure 2.1: A metric on the noose space for which the span and semispan equal 1, and the surjective span and surjective semispan equal $\frac{1}{4}$. The vertical line near the center of the picture is identified with the left edge of the picture, thus making each horizontal cross-section homeomorphic to the noose space; likewise, the horizontal line near the middle is identified with the bottom edge of the picture. A pixel is colored black if the corresponding pair of points in the noose are at distance 0 (so the set of black points is precisely the diagonal); dark gray indicates a distance of $\frac{1}{4}$; medium gray indicates a distance of $\frac{1}{2}$; light gray indicates a distance of $\frac{3}{4}$; white indicates a distance of 1.

Chapter 3

A non-chainable continuum with span zero

3.1 Definitions and notation

Recall that given a continuum X, the *span of* X is the supremum of all $\eta \geq 0$ for which there exists a subcontinuum Z of $X \times X$ such that: 1) $d(x,y) \geq \eta$ for each $(x,y) \in Z$; and 2) $\pi_1(Z) = \pi_2(Z)$, where $\pi_1, \pi_2 : X \times X \to X$ are the first and second coordinate projections, respectively.

The following properties are straightforward (see [23]):

- if X and Y are continua with $X \subseteq Y$, then $\mathrm{span}(X) \leq \mathrm{span}(Y)$;
- the arc $[0,1]$ has span zero; and
- if $\langle X_n \rangle_{n=1}^{\infty}$ is a sequence of continua in a given compact metric space, then

$$\limsup_{n\to\infty} \mathrm{span}(X_n) \leq \mathrm{span}(\limsup_{n\to\infty} X_n).$$

The third property implies in particular that given any continuum $X \subset \mathbb{R}^2$ and any $\varepsilon > 0$, there is some $\delta > 0$ such that $\mathrm{span}(\overline{X_\delta}) < \mathrm{span}(X) + \varepsilon$, where X_δ denotes the δ-neighborhood of X in $\mathbb{R}^2$.

A *simple triod* is a continuum T which is the union of three arcs, A_1, A_2, A_3, which have a common endpoint o and are otherwise pairwise disjoint. A_1, A_2, A_3 are called the *legs* of T, and o is the *branch point* of T.

If $f : X \to Y$ is a function and $x_1, \ldots, x_n \in X$, we will often write

$$x_1 \cdots x_n \overset{f}{\mapsto} y_1 \cdots y_n$$

to mean $f(x_i) = y_i$ for each i.

Given a set S, a *total quasi-order on S* is a binary relation $\leq$ on S which is reflexive and transitive, and which satisfies the property that for every $s_1, s_2 \in S$, we have $s_1 \leq s_2$ or $s_2 \leq s_1$ (or both). If $\leq$ is a total quasi-order, we write $s_1 \simeq s_2$ to mean $s_1 \leq s_2$ and $s_2 \leq s_1$, and we write $s_1 < s_2$ to mean $s_1 \leq s_2$ and $s_2 \not\leq s_1$. Elements s_1 and s_2 are *$\leq$-adjacent* if $s_1 \not\simeq s_2$ and there is no $s \in S$ with $s_1 < s < s_2$ or $s_2 < s < s_1$.

If S is finite and $\leq$ is a total quasi-order on S, then there is a function $f : S \to \mathbb{Z}$ which is order preserving (i.e. $f(s_1) \leq f(s_2)$ if and only if $s_1 \leq s_2$) whose range is a contiguous block of integers.

By a *graph*, we will mean an undirected connected graph without multiple edges joining the same pair of vertices, and without any edge from a vertex to itself. If G is a graph, $V(G)$ denotes the set of vertices. A pair of vertices $v_1, v_2 \in V(G)$ is *adjacent in G* provided there is an edge between them. A sequence of distinct vertices $v_1, \ldots, v_n \in V(G)$ is *consecutive in G* provided there is an edge between v_i and v_{i+1} for each $0 \leq i \leq n-1$.

A graph G will be considered as a topological space in the usual way, where the edges are realized by arcs. If $v_1, v_2 \in V(G)$ are adjacent in G, then we will use the notation $[v_1, v_2]_G$ to denote the arc joining v_1 and v_2; we will often drop the subscript G when the graph is clear from the context.

A *tree* is a continuum-theoretic graph (i.e. a connected finite union of arcs with pairwise finite intersections) which contains no circle. If T is a tree and $a, b \in T$, then $[a, b]_T$ denotes the minimal arc $A \subseteq T$ with $a, b \in A$; again, we will often drop the subscript T when the tree is clear from the context.

By a *word*, we will mean a finite sequence of symbols from some alphabet Γ, where an *alphabet* is any set of symbols not including (,), [,], $\prod$, $\pitchfork$, and $^{\leftarrow}$. If ω is a word, then $|\omega|$ denotes the length of ω. A word ω will be considered as a function on the set of integers $\{0, 1, \ldots, |\omega| - 1\}$. $\omega^{\leftarrow}$ denotes the reverse of ω, defined by $\omega^{\leftarrow}(j) = \omega(|\omega| - j - 1)$.

If $\omega_1, \ldots, \omega_n$ is a sequence of words, then $\prod_{i=1}^{n} \omega_i$ denotes the concatenation of these words. If ω is a word and n is a non-negative integer, then ω^n denotes the word obtained by repeating ω n times; that is, $\omega^n = \prod_{i=1}^{n} \omega$. Given words ω_1, ω_2 such that the last symbol of ω_1 coincides with the first symbol of ω_2, define $\omega_1 \pitchfork \omega_2$ to be the word obtained by concatenating onto ω_1 all but the first symbol in ω_2. For example, $abc \pitchfork caba = abcaba$.

3.2 Graph-words

3.2.1 Sketches and the graph-word ρ_N

Definition. A *graph-word in the alphabet* Γ is a pair $\rho = \langle G_\rho, w_\rho \rangle$ where G_ρ is a graph, and $w_\rho : V(G_\rho) \to \Gamma$ is a function.

Let us fix, for the rest of this paper, the alphabet $\Gamma := \{a, b, c\} \cup \{b_t : t \in [0, 1]\}$.

For each positive integer N, denote by α_N, β_N, γ_N the following three words:

$$(abc)^{2N+1} \prod_{i=0}^{2N-1} [ab_{i/2N}cb_{i/2N}a(cba)^{2N-i-1}cbc(abc)^{2N-i-1}]\ ab_1cb_1a(cba)^{2N+1}$$

$$(abc)^{2N+1} \prod_{i=0}^{2N-1} [ab_{i/2N}cb_{i/2N}a(cba)^{2N-i-1}cbabc(abc)^{2N-i-1}]\ ab_1cb_1a(cba)^{2N+1}cb$$

$$ac$$

For later use, we also define the word β_N^- to be identical to the word β_N except without the final b.

Define the graph-word ρ_N as follows. Let G_{ρ_N} be a simple triod, with vertex set $V(G_{\rho_N}) = \{o, p_1, \ldots, p_{|\alpha_N|-1}, q_1, \ldots, q_{|\beta_N|-1}, r\}$, where o is the branch point of the triod, $p_{|\alpha_N|-1}, q_{|\beta_N|-1}, r$ are the endpoints of G_{ρ_N}, the points p_j belong to the leg $[o, p_{|\alpha_N|-1}]$ with $p_j \in [o, p_{j+1}]$ for each j, and the points q_j belong to the leg $[o, q_{|\beta_N|-1}]$ with $q_j \in [o, q_{j+1}]$ for each j. Put $p_0 := o$ and $q_0 := o$. Define $w_{\rho_N} : V(G_{\rho_N}) \to \Gamma$ by $w_{\rho_N}(p_j) := \alpha_N(j)$, $w_{\rho_N}(q_j) := \beta_N(j)$, and $w_{\rho_N}(r) := \gamma_N(1) = c$.

To construct the example of a non-chainable continuum X with span zero, we will define a sequence of simple triods $\langle T_N \rangle_{N=0}^{\infty}$ in the plane such that T_N is contained in a small neighborhood of T_{N-1} in $\mathbb{R}^2$ for each $N > 0$; X will then be defined as the intersection of the nested sequence of neighborhoods of the triods T_N. The graph-word ρ_N will be used to describe the pattern with which we nest the simple triod T_N inside a small neighborhood of T_{N-1}. To carry this out precisely, we introduce the notion of a *sketch* below.

Remark. The space X may alternatively be described as an inverse limit of simple triods, as follows. Let T be a simple tirod with endpoints denoted as a, b, c and branch point o. Denote a point in the interior of the arc $[o, b]$ by b_0, and parameterize the arc $[b_0, b]$ by b_t for $t \in [0, 1]$, so that $b_1 = b$ (as per the notion of a Γ-marking defined below). Then the N-th bonding map $b_N : T \to T$ takes o to a, is the identity on the segment

$[b_0, b]$, and otherwise maps the legs $[o, a]$, $[o, b]$, $[o, c]$ in a piecewise linear way according to the patterns α_N, β_N, γ_N, respectively. Figures 3.1, 3.2, and 3.3, along with the proof of Proposition 5 below, provide some geometric intuition for how this looks.

Definition. Given a simple triod T with branch point o, a Γ-*marking of* T is a function $\iota : \Gamma \to T$ such that $\iota(a)$, $\iota(b)$, $\iota(c)$ are the endpoints of T and $\{\iota(b_t) : t \in [0, 1]\} \subset [o, \iota(b)]$ are such that whenever $t < t'$, we have $\iota(b_t) \in [o, \iota(b_{t'})]$ and $\mathrm{diam}([\iota(b_t), \iota(b_{t'})]) = d(\iota(b_t), \iota(b_{t'})) = t' - t$.

Define the simple triod $T_0 := \{(x, 0) : x \in [-1, 1]\} \cup \{(0, y) : y \in [0, 2]\} \subset \mathbb{R}^2$, and define a Γ-marking $\iota : \Gamma \to T_0$ by:

$$\begin{aligned}
\iota(a) &:= (-1, 0) \\
\iota(b) &:= (0, 2) \\
\iota(c) &:= (1, 0) \\
\iota(b_t) &:= (0, 1 + t) \text{ for } t \in [0, 1].
\end{aligned}$$

Definition. Define the equivalence relation $\approx_\Gamma$ on Γ by $\sigma \approx_\Gamma \tau$ if and only if $\sigma = \tau$ or $\sigma, \tau \in \{b\} \cup \{b_t : t \in [0, 1]\}$.

The relation $\approx_\Gamma$ partitions Γ into three equivalence classes. If ι is a Γ-marking of a triod T, then $\sigma \approx_\Gamma \tau$ if and only if $\iota(\sigma)$ and $\iota(\tau)$ belong to the same leg of T.

To simplify definitions and arguments in the following, we will restrict our attention to a special class of graph-words.

Definition. A *compliant graph-word* is a graph-word $\langle G, w \rangle$ in the alphabet Γ such that there is no pair of adjacent vertices v_1, v_2 in G with $w(v_1) \approx_\Gamma w(v_2)$.

Observe that ρ_N is a compliant graph-word for each N.

Definition. Suppose T is a simple triod with a Γ-marking $\iota : \Gamma \to T$, and let $\rho = \langle G, w \rangle$ be a compliant graph-word in the alphabet Γ. Then $\widehat{w} : G \to T$ is a ρ-*suggested bonding map* provided $\widehat{w}|_{V(G)} = \iota \circ w$, and for any adjacent $v_1, v_2 \in V(G)$, we have that $\widehat{w}|_{[v_1, v_2]_G}$ is a homeomorphism from $[v_1, v_2]_G$ to $[\iota(w(v_1)), \iota(w(v_2))]_T$.

Definition. Let $\langle \Omega, d \rangle$ be a metric space, let $T \subseteq \Omega$ be a Γ-marked simple triod, let $G \subseteq \Omega$ be a graph, and let $\varepsilon > 0$. Then $\rho = \langle G, w \rangle$ is a $\langle T, \varepsilon \rangle$-*sketch of* G *in* Ω if ρ is a compliant graph-word in the alphabet Γ, and there is a ρ-suggested bonding map $\widehat{w} : G \to T$ such that $d(x, \widehat{w}(x)) < \frac{\varepsilon}{2}$ for every $x \in G$.

The next proposition assures us that we may use the graph word ρ_N defined above to describe the pattern with which we embed one simple triod into a small neighborhood of another, in the plane.

We will need some additional notation when working with the graph-word ρ_N. For each $i \leq 2N$, define $n(i)$ and $m(i)$ to be the unique integers such that

$$(n(i)-1)\ n(i)\ (n(i)+1) \overset{\alpha_N}{\mapsto} b_{i/2N} c b_{i/2N}$$
$$(m(i)-1)\ m(i)\ (m(i)+1) \overset{\beta_N}{\mapsto} b_{i/2N} c b_{i/2N}.$$

For each $i < 2N$, define $\theta(i) := 6N - 3i + 1$ and $\phi(i) := 6N - 3i + 2$. Observe that

$$(n(i)+\theta(i)-1)\ (n(i)+\theta(i))\ (n(i)+\theta(i)+1) \overset{\alpha_N}{\mapsto} cbc$$
$$(m(i)+\phi(i)-2)\ (m(i)+\phi(i)-1) \cdots (m(i)+\phi(i)+2) \overset{\beta_N}{\mapsto} cbabc.$$

Note that $n(0) = m(0) = 6N+5$, and that $n(i)+2\theta(i) = n(i+1)$ and $m(i)+2\phi(i) = m(i+1)$ for each $i < 2N$.

When discussing an embedding of the graph G_{ρ_N} in $\mathbb{R}^2$, we will make no notational distinction between the points of G_{ρ_N} before and after the embedding.

Proposition 5. *Suppose $T \subset \mathbb{R}^2$ is a simple triod and $\iota : \Gamma \to T$ is a Γ-marking. For any integer $N > 0$ and any $\varepsilon > 0$, there is an embedding of the simple triod graph G_{ρ_N} in $\mathbb{R}^2$ such that ρ_N is a $\langle T, \varepsilon\rangle$-sketch of G_{ρ_N} in $\mathbb{R}^2$. Moreover, the embedding can be chosen such that $q_{|\beta_N|-1} = \iota(b)$, $q_{|\beta_N|-2} = \iota(c)$, and $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{G_{\rho_N}} = [\iota(c), \iota(b)]_T$.*

Proof. For simplicity, we will argue only the case $T = T_0$, with the Γ-marking ι as described above; the general case can be treated similarly.

First we will analytically define a different embedding of G_{ρ_N} in $\mathbb{R}^2$, then we will describe how to obtain the desired embedding from it.

Let $\eta > 0$ be significantly smaller than ε, say $\eta < \frac{\varepsilon}{20N^2}$. For $0 \leq i \leq 2N$, put

$$p_{n(i)} := (1+\eta, (4i+\tfrac{3}{2})\eta),$$
$$q_{m(i)} := (1, (4i+\tfrac{3}{2})\eta).$$

For $0 \leq i < 2N$ and $1 \leq j < \theta(i)$, put

$$p_{n(i)+j} := (1-j, (4i+3)\eta),$$
$$p_{n(i+1)-j} := (1-j, 4(i+1)\eta),$$

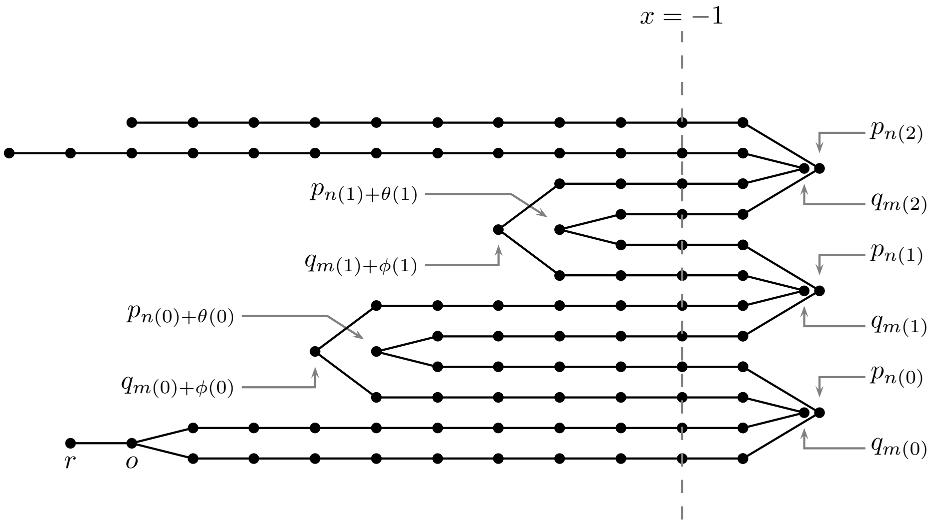


Figure 3.1: The intermediate stage G' for the embedding of G_{ρ_1} in $\mathbb{R}^2$.

and put $p_{n(i)+\theta(i)} := (1-\theta(i), (4i+\frac{7}{2})\eta)$. For $0 \le i < 2N$ and $1 \le j < \phi(i)$, put

$$q_{m(i)+j} := (1-j, (4i+2)\eta),$$
$$q_{m(i+1)-j} := (1-j, (4(i+1)+1)\eta),$$

and put $q_{m(i)+\phi(i)} := (1-\phi(i), (4i+\frac{7}{2})\eta)$. Further, put

$$\begin{aligned} p_{n(0)-j} &:= (1-j, 0) && \text{for } 1 \le j < 6N+5, \\ q_{m(0)-j} &:= (1-j, \eta) && \text{for } 1 \le j < 6N+5, \\ q_{m(2N)+j} &:= (1-j, (8N+2)\eta) && \text{for } 1 \le j \le 6N+7, \\ p_{n(2N)+j} &:= (1-j, (8N+3)\eta) && \text{for } 1 \le j \le 6N+5. \end{aligned}$$

Finally, put $o := (-6N-4, \frac{1}{2}\eta)$ and $r := (-6N-5, \frac{1}{2}\eta)$. Join each pair of adjacent vertices in G_{ρ_N} by a straight line segment in $\mathbb{R}^2$. Denote the resultant embedding of G_{ρ_N} in $\mathbb{R}^2$ by G'. Figure 3.1 depicts the embedding G' for $N = 1$.

Observe that in G', for each integer $k \le -1$, if v and v' are two vertices in the line $x = k$, then $w(v) = w(v')$. Also notice that each vertex v in the line $x = -1$ is already close to the point $\iota(w(v)) = \iota(a) = (-1, 0)$, and that each vertex u of the form $p_{n(i)}$ or $q_{m(i)}$ is already close to the point $\iota(w(u)) = \iota(c) = (1, 0)$. We now describe heuristically in two steps how to mold G' into the embedding we seek.

First, for each $i \le 2N$, for each triple $\langle v_1, v_2, v_3\rangle$ of the form $\langle p_{n(i)-2}, p_{n(i)-1}, p_{n(i)}\rangle$, $\langle q_{m(i)-2}, q_{m(i)-1}, q_{m(i)}\rangle$, $\langle p_{n(i)+2}, p_{n(i)+1}, p_{n(i)}\rangle$, or $\langle q_{m(i)+2}, q_{m(i)+1}, q_{m(i)}\rangle$, move the vertex v_2

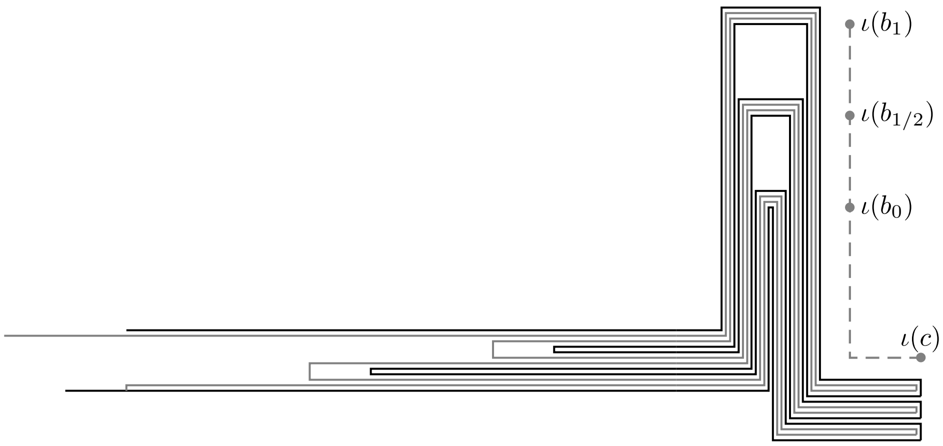


Figure 3.2: The second intermediate stage for the embedding of G_{ρ_1} in $\mathbb{R}^2$.

up to be close to the point $\iota(b_{i/2N})$, move the vertex v_3 down slightly, and shape the arcs joining v_1 to v_2 and v_2 to v_3 so that:

(1) there is a homeomorphism $\hat{w}_1 : [v_1, v_2]_{G'} \to [\iota(a), \iota(b_{i/2N})]_{T_0}$ such that $\hat{w}_1(v_1) = \iota(a)$, $\hat{w}_1(v_2) = \iota(b_{i/2N})$, and $d(x, \hat{w}_1(x)) < \eta$ for each $x \in [v_1, v_2]_{G'}$,

(2) there is a homeomorphism $\hat{w}_2 : [v_2, v_3]_{G'} \to [\iota(b_{i/2N}), \iota(c)]_{T_0}$ such that $\hat{w}_2(v_2) = \iota(b_{i/2N})$, $\hat{w}_2(v_3) = \iota(c)$, and $d(x, \hat{w}_2(x)) < \eta$ for each $x \in [v_2, v_3]_{G'}$, and

(3) $[v_1, v_2]_{G'} \cup [v_2, v_3]_{G'}$ misses the closed upper-right quandrant of the plane $\{(x, y) : x \geq 0, y \geq 0\}$,

and so that in the end no new intersections between those arcs have been introduced (i.e., so that the result is still an embedding of G_{ρ_N}). Figure 3.2 depicts the result for $N = 1$.

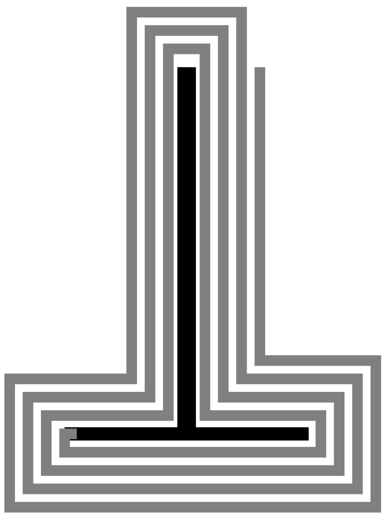
Next, take the strip $\{(x, y) : x \leq -1, 0 \leq y \leq (8N + 3)\eta\}$ and stretch and wind it counter-clockwise $2N + 2$ times around the outside of

$$\bigcup_{i=0}^{2N} \left([p_{n(i)-2}, p_{n(i)+2}]_{G'} \cup [q_{m(i)-2}, q_{m(i)+2}]_{G'} \right),$$

so that for each integer $k \leq -1$, all the vertices v in the line $x = k$ end up near the point $\iota(w(v)) \in T_0$, taking care to make sure $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{G'} = [\iota(c), \iota(b)]_{T_0}$. Figure 3.3 depicts roughly how this wrapping looks.

The resulting embedding satisfies the desired properties. □

(a) Before wrapping



(b) After wrapping

Figure 3.3: Wrapping the strip counterclockwise around the simple triod to obtain the embedding of G_{ρ_N} in $\mathbb{R}^2$.

3.2.2 Span and ρ_N

In this section we prove that the span of a simple triod embedded in a way described by ρ_N converges to 0 as $N \to \infty$. This will ensure that we will obtain a continuum with span zero when we take the nested intersection of neighborhoods of triods embedded in $\mathbb{R}^2$ as described by the ρ_N's.

Given a set T and a subset $S \subseteq T \times T$, let $S^{-1} = \{(x,y) \in T \times T : (y,x) \in S\}$.

Lemma 6. *Let T be a simple triod with legs A_1, A_2, A_3 and branch point o. For each $i \in \{1,2,3\}$ let p_i be the endpoint of leg A_i other than o. Suppose $\delta > 0$ and $W \subset A_1 \times A_2$ is an arc such that $(o,o) \in W$, W meets $(\{p_1\} \times A_2) \cup (A_1 \times \{p_2\})$, and $d(x_1,x_2) \leq \delta$ for each $(x_1,x_2) \in W$. Then the span of T is $\leq \delta$.*

Proof. Suppose $Z \subset T \times T$ is a subcontinuum with $\pi_1(Z) = \pi_2(Z)$. If $\pi_1(Z)$ is an arc, then it is easy to see that Z meets the diagonal $\Delta T = \{(x,x) : x \in T\}$, as arcs have span zero.

If $\pi_1(Z)$ is a subtriod T' of T, then we may assume $T = T'$ by replacing the arc W by the component of $W \cap (T' \times T')$ that contains (o,o). Let K_1 and K_2 be disjoint closed and open subsets of $(A_1 \times A_2) \smallsetminus W$ such that $(A_1 \times \{o\}) \smallsetminus W \subset K_1$, $(\{o\} \times A_2) \smallsetminus W \subset K_2$, and $K_1 \cup K_2 = (A_1 \times A_2) \smallsetminus W$.

For each $i \in \{1,2,3\}$ let U_i and V_i be the two components of $(A_i \times A_i) \smallsetminus \Delta T$, where $(A_i \smallsetminus \{o\}) \times \{o\} \subset U_i$ and $\{o\} \times (A_i \smallsetminus \{o\}) \subset V_i$. It can then be seen that the set

$$Y := \big(U_1 \cup U_2 \cup V_3 \cup (A_1 \times A_3) \cup (A_2 \times A_3) \cup K_1 \cup K_2^{-1}\big) \smallsetminus W$$

is closed and open in $(T \times T) \smallsetminus (W \cup W^{-1} \cup \Delta T)$ (see Proposition 5.1 of [13]). Note that $(T \times T) \smallsetminus (W \cup W^{-1} \cup \Delta T) = Y \cup Y^{-1}$.

Observe that $p_3 \notin \pi_1(Y)$ and $p_3 \notin \pi_2(Y^{-1})$, hence $Z \not\subseteq Y$ and $Z \not\subseteq Y^{-1}$. Since Z is connected, it follows that Z must meet $W \cup W^{-1} \cup \Delta T$.

Thus in either case, there is some $(x_1,x_2) \in Z$ with $d(x_1,x_2) \leq \delta$. Therefore T has span $\leq \delta$. □

Proposition 7. *Suppose $T \subset \mathbb{R}^2$ is Γ-marked. If the triod graph G_{ρ_N} is embedded in $\mathbb{R}^2$ such that ρ_N is a $\langle T, \varepsilon \rangle$-sketch of G_{ρ_N} in $\mathbb{R}^2$, then the span of G_{ρ_N} is less than $\frac{1}{2N} + \varepsilon$.*

Proof. In order to apply Lemma 6, we will produce an arc $W \subset [o, p_{|\alpha_N|-1}] \times [o, q_{|\beta_N|-1}]$. Intuitively, one may think of W as a pair of points travelling simultaneously, one on the leg $[o, p_{|\alpha_N|-1}]$ and the other on $[o, q_{|\beta_N|-1}]$, starting with both at o, ending with one at the end of its leg, and at every moment staying within distance $\frac{1}{2N} + \varepsilon$ from one another. With this in mind, and referring to Figure 3.1, one should be easily convinced that such a W may be defined which passes through the following pairs, in order: (o, o), $(p_{n(0)}, q_{m(0)})$, $(p_{n(0)-\phi(0)}, q_{m(0)+\phi(0)})$, $(p_{n(0)}, q_{m(1)})$, $(p_{n(0)+\theta(0)}, q_{m(1)-\theta(0)})$, $(p_{n(1)}, q_{m(1)})$, $\ldots$, $(p_{n(2N)}, q_{m(2N)})$, $(p_{|\alpha_N|-1}, q_{m(2N)+6N+5})$. The precise definition of this arc W follows.

Suppose that n, n' and m, m' are two pairs of adjacent integers. Let $S^{n,n'}_{m,m'}$ denote the square $[p_n, p_{n'}] \times [q_m, q_{m'}]$. Suppose one of the following occurs:

(1) $w(p_n) = w(q_m)$, $w(p_{n'}) = w(q_{m'})$;

(2) $w(p_n) = w(q_m)$, $w(p_{n'}) = b_{i/2N}$, $w(q_{m'}) = b_{(i+1)/2N}$ for some i; or

(3) $w(p_{n'}) = w(q_{m'})$, $w(p_n) = b_{i/2N}$, $w(q_m) = b_{(i+1)/2N}$ for some i.

Then let $W^{n,n'}_{m,m'} \subset S^{n,n'}_{m,m'}$ be an arc such that $d(x_1, x_2) < \frac{1}{2N} + \varepsilon$ for each $(x_1, x_2) \in W^{n,n'}_{m,m'}$, and $W^{n,n'}_{m,m'} \cap \partial S^{n,n'}_{m,m'} = \{(p_n, q_m), (p_{n'}, q_{m'})\}$.

Define the arc $W \subset [o, p_{|\alpha_N|-1}] \times [o, q_{|\beta_N|-1}]$ as follows. It will be helpful to refer to Figure 3.1 when reading this formula.

$$\begin{aligned} W := \bigcup_{j=0}^{n(0)-1} W^{j,j+1}_{j,j+1} \cup \bigcup_{i=0}^{2N-1} \Bigg(& \bigcup_{j=0}^{\phi(i)-1} W^{n(i)-j,\ n(i)-j-1}_{m(i)+j,\ m(i)+j+1} \cup \\ & \bigcup_{j=0}^{\phi(i)-1} W^{n(i)-\phi(i)+j,\ n(i)-\phi(i)+j+1}_{m(i)+\phi(i)+j,\ m(i)+\phi(i)+j+1} \cup \\ & \bigcup_{j=0}^{\theta(i)-1} W^{n(i)+j,\ n(i)+j+1}_{m(i+1)-j,\ m(i+1)-j-1} \cup \\ & \bigcup_{j=0}^{\theta(i)-1} W^{n(i)+\theta(i)+j,\ n(i)+\theta(i)+j+1}_{m(i+1)-\theta(i)+j,\ m(i+1)-\theta(i)+j+1} \Bigg) \cup \\ & \bigcup_{j=0}^{6N+4} W^{n(2N)+j,\ n(2N)+j+1}_{m(2N)+j,\ m(2N)+j+1}. \end{aligned}$$

Then W contains (o, o) and meets $\{p_{|\alpha_N|-1}\} \times [o, q_{|\beta_N|-1}]$, and $d(x_1, x_2) < \frac{1}{2N} + \varepsilon$ for each $(x_1, x_2) \in W$, hence the claim follows by Lemma 6. □

3.3 Combinatorics from chain covers

3.3.1 Chain quasi-orders

The following definition is closely related to the notion of a chain word reduction from [32]. It should be thought of as follows: if $\langle G, w\rangle$ is a $\langle T, \varepsilon\rangle$-sketch of G and we have a chain cover of G of small mesh, then $v_1 \leq v_2$ means roughly that the chain "covers v_1 before, or at around the same time as, v_2" (see Proposition 9).

Definition. Suppose $\langle G, w\rangle$ is a compliant graph-word. A *chain quasi-order of* $\langle G, w\rangle$ is a total quasi-order $\leq$ on $V(G)$ satisfying:

(C1) if $v_1 \simeq v_2$, then $w(v_1) \approx_\Gamma w(v_2)$;

(C2) if $v_1, v_2 \in V(G)$ are adjacent in G, then v_1 and v_2 are $\leq$-adjacent; and

(C3) if $v_1, v_2, v_3 \in V(G)$ are consecutive in G, $v \in V(G)$, and if $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0, 1]$ are such that $t' \geq t$, $v_1v_2v_3 \overset{w}{\mapsto} \sigma b_t \tau$, $w(v) = b_{t'}$, and $v_1 < v_2 \simeq v < v_3$, then $t' - t < \frac{1}{2}$.

Notice that if $\leq$ is a chain quasi-order, then the reverse order of $\leq$ (defined by $v_1 \leq^* v_2$ if and only if $v_2 \leq v_1$) is also a chain quasi-order.

The following simple lemma will be useful later on.

Lemma 8. *Let $\leq$ be a chain quasi-order of $\langle G, w\rangle$. Suppose $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in G and $v \in V(G)$ is such that $v_1 < v < v_2$. Then there is some $i \in \{1, \ldots, \kappa\}$ such that $v \simeq s_i$.*

Proof. Put $s_0 := v_1$, $s_{\kappa+1} := v_2$, and let i be the largest integer in $\{0, \ldots, \kappa\}$ such that $s_i \leq v$. Then $s_{i+1} > v$, so since s_i and s_{i+1} are $\leq$-adjacent by property (C2), we must have $s_i \geq v$. Thus $s_i \simeq v$. □

3.3.2 Chain covers and the triod T_0

Proposition 9. *Suppose $\langle G, w\rangle$ is a compliant graph-word which is a $\langle T_0, \varepsilon\rangle$-sketch of a graph G in $\mathbb{R}^2$, where $0 < \varepsilon < \frac{1}{2}$. If there is a chain cover for G of mesh $< \frac{1}{2} - \varepsilon$, then there is a chain quasi-order of $\langle G, w\rangle$.*

Proof. Let $\mathcal{U} = \langle U_\ell : 0 \leq \ell < L \rangle$ be a chain cover for G of mesh $< \frac{1}{2} - \varepsilon$, ordered so that $U_\ell \cap U_{\ell'} \neq \emptyset$ if and only if $|\ell - \ell'| \leq 1$. For each $v \in V(G)$, let $\ell(v)$ be such that $v \in U_{\ell(v)}$ (for each v there are either one or two choices for $\ell(v)$).

Observe that if $v_1, v_2 \in V(G)$ and $\ell(v_1) = \ell(v_2)$, then $w(v_1) \approx_\Gamma w(v_2)$, since otherwise $d(\iota(w(v_1)), \iota(w(v_2))) \geq \sqrt{2} > \frac{1}{2}$, hence $d(v_1, v_2) > \frac{1}{2} - \varepsilon$, contradicting the condition that the diameter of $U_{\ell(v_1)} = U_{\ell(v_2)}$ is $< \frac{1}{2} - \varepsilon$.

Define the relation $\leq$ on $V(G)$ by setting $v_1 \leq v_2$ if and only if for every $v \in V(G)$ satisfying $\ell(v_2) \leq \ell(v) \leq \ell(v_1)$ we have $w(v) \approx_\Gamma w(v_1)$.

The following properties follow directly from the definition of $\leq$:

Properties. (1) If $\ell(v_1) \leq \ell(v_2)$, then $v_1 \leq v_2$.

(2) If $v_1 \leq v_2$ and $w(v_1) \not\approx_\Gamma w(v_2)$, then $\ell(v_1) < \ell(v_2)$.

(3) If $v_1, v_2 \in V(G)$ are $\leq$-adjacent, then $w(v_1) \not\approx_\Gamma w(v_2)$.

It is straightforward to check using the definition and these properties that $\leq$ is a total quasi-order.

We now check that $\leq$ satisfies properties (C1), (C2), and (C3) of the definition of a chain quasi-order.

(C1): Suppose $v_1, v_2 \in V(G)$ with $v_1 \simeq v_2$. Assume without loss of generality that $\ell(v_2) \leq \ell(v_1)$. It then follows immediately from the definition of $\leq$ and the assumption $v_1 \leq v_2$ that $w(v_1) \approx_\Gamma w(v_2)$ (take $v = v_2$).

(C2): Suppose $v_1, v_2 \in V(G)$ are adjacent in G. Since $\langle G, w \rangle$ is compliant, we know that $w(v_1) \not\approx_\Gamma w(v_2)$. Let $\sigma := w(v_1)$ and $\tau := w(v_2)$. Assume without loss of generality that $\ell(v_1) < \ell(v_2)$, which implies that $v_1 < v_2$.

If $v \in V(G)$ were such that $w(v) \not\approx_\Gamma \sigma, \tau$ and $v_1 < v < v_2$, then $\ell(v_1) < \ell(v) < \ell(v_2)$. This would imply that the link $U_{l(v)}$ contains the point v and meets the arc $[v_1, v_2]$. Since $\langle G, w \rangle$ is compliant, the only possible cases are:

$$\{\sigma, \tau\} = \{a, b\} \text{ and } w(v) = c$$
$$\{\sigma, \tau\} = \{a, c\} \text{ and } w(v) \in \{b\} \cup \{b_t : t \in [0,1]\}$$
$$\{\sigma, \tau\} = \{b, c\} \text{ and } w(v) = a$$
$$\{\sigma, \tau\} = \{a, b_t\} \text{ and } w(v) = c \qquad \text{(for some } t \in [0,1])$$
$$\{\sigma, \tau\} = \{c, b_t\} \text{ and } w(v) = a \qquad \text{(for some } t \in [0,1])$$

In each case, we have $d(\iota(w(v)), [\iota(\sigma), \iota(\tau)]) \geq 1 > \frac{1}{2}$. But this yields a contradiction, since $\mathcal{U}$ has mesh $< \frac{1}{2} - \varepsilon$.

Suppose for a contradiction that v_1, v_2 are not adjacent in the $\leq$ order. Let v, v' be such that $v_1 < v < v'$, and v_1, v are $\leq$-adjacent and v, v' are $\leq$-adjacent. By the above, we have that $w(v), w(v')$ are each $\approx_\Gamma$ to either σ or τ, hence by Property (3) the only possibility is $w(v) \approx_\Gamma \tau$, $w(v') \approx_\Gamma \sigma$. Property (2) then implies that $\ell(v_1) < \ell(v) < \ell(v')$.

Define the arc $A \subset T_0$ according to the value of σ as follows:

$$A := \begin{cases} [\iota(a), o] & \text{if } \sigma = a \\ [\iota(c), o] & \text{if } \sigma = c \\ [\iota(b), o] & \text{if } \sigma \in \{b\} \cup \{b_t : t \in [0,1]\} \end{cases}.$$

In each case, observe that $d(\iota(w(v)), A) \geq 1 > \frac{1}{2}$, and also $B_{\frac{1}{2}}(\iota(\sigma)) \subset A$ and $B_{\frac{1}{2}}(\iota(w(v'))) \subset A$.

Applying the homeomorphism $\widehat{w}|_{[v_1,v_2]}$ yields the chain cover $\langle \widehat{w}(U_\ell \cap [v_1, v_2]) : \ell' \leq \ell \leq \ell'' \rangle$ of the arc $[\iota(\sigma), \iota(\tau)]_{T_0}$, where $\ell' := \min\{\ell : U_\ell \cap [v_1, v_2] \neq \emptyset\}$ and $\ell'' := \max\{\ell : U_\ell \cap [v_1, v_2] \neq \emptyset\}$.

Notice that $\widehat{w}(U_{\ell(v_1)})$ and $\widehat{w}(U_{\ell(v')})$ are sets of diameter $< \frac{1}{2}$ containing $\iota(\sigma)$ and $\iota(w(v'))$, respectively, hence are subsets of A. It follows in particular that the links $\widehat{w}(U_{\ell(v_1)} \cap [v_1, v_2])$ and $\widehat{w}(U_{\ell(v')} \cap [v_1, v_2])$ both meet the arc $A \cap [\iota(\sigma), \iota(\tau)]_{T_0}$, which implies each link $\widehat{w}(U_\ell \cap [v_1, v_2])$, $\ell(v_1) < \ell < \ell(v')$, must meet A as well. But $\widehat{w}(U_{\ell(v)})$ has diameter $< \frac{1}{2}$ and contains $\iota(w(v))$, hence misses A by the above. This is a contradiction, therefore we must have that v_1 and v_2 are $\leq$-adjacent.

(C3): Suppose $v \in V(G)$, v_1, v_2, v_3 are consecutive in G, and that $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0,1]$ are such that $t' \geq t$, $w(v) = b_{t'}$, $v_1 v_2 v_3 \overset{w}{\mapsto} \sigma b_t \tau$, and $v_1 < v_2 \simeq v < v_3$.

From Property (2) we know that $\ell(v)$ is between $\ell(v_1)$ and $\ell(v_3)$, hence the link $U_{\ell(v)}$ contains v and meets the arc $[v_1, v_2] \cup [v_2, v_3]$. Since $d(\iota(b_{t'}), [\iota(\sigma), \iota(b_t)]_{T_0} \cup [\iota(b_t), \iota(\tau)]_{T_0}) = d(\iota(b_{t'}), \iota(b_t)) = t' - t$ and $\mathcal{U}$ has mesh $< \frac{1}{2} - \varepsilon$, it follows that $t' - t < \frac{1}{2}$. □

3.4 Combinatorics of the graph-word ρ_N

3.4.1 Chain quasi-orders and ρ_N

Throughout this subsection assume that $\langle G, w \rangle$ is a compliant graph-word, and that $\leq$ is a chain quasi-order of $\langle G, w \rangle$.

Let $f : V(G) \to \mathbb{Z}$ be an order preserving function whose range is a contiguous block of integers.

Lemma 10. *Suppose $v_1, \ldots, v_n$ are consecutive in G, and that for each $1 < j < n$ we have $w(v_{j-1}) \not\approx_\Gamma w(v_{j+1})$. Then $f(v_1), \ldots f(v_n)$ are consecutive integers, i.e. either $f(v_{j+1}) = f(v_j) + 1$ for each $1 \leq j < n$, or $f(v_{j+1}) = f(v_j) - 1$ for each $1 \leq j < n$.*

Proof. This follows immediately from properties (C1) and (C2) of the chain quasi-order $\leq$. □

As an application of Lemma 10, we make the following observation.

Lemma 11. *Suppose for some $i < 2N$ that $v_0, v_1, \ldots, v_{2\theta(i)} \in V(G)$ are consecutive in G with $v_0 \cdots v_{2\theta(i)} \overset{w}{\mapsto} \alpha_N(n(i)) \cdots \alpha_N(n(i+1))$. Let $k := f(v_0)$. Then we have one of the following four cases:*

(1) $v_0 \cdots v_{2\theta(i)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i))$;

(2) $v_0 \cdots v_{\theta(i)} \overset{f}{\mapsto} k \cdots (k + \theta(i))$, $v_{\theta(i)} \cdots v_{2\theta(i)} \overset{f}{\mapsto} (k + \theta(i)) \cdots k$;

(3) $v_0 \cdots v_{\theta(i)} \overset{f}{\mapsto} k \cdots (k - \theta(i))$, $v_{\theta(i)} \cdots v_{2\theta(i)} \overset{f}{\mapsto} (k - \theta(i)) \cdots k$; or

(4) $v_0 \cdots v_{2\theta(i)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i))$.

Moreover, the analogous statement holds for the word β_N^- (where we replace n with m and θ with ϕ).

Proof. This is a simple consequence of Lemma 10. □

Lemma 12. *Suppose that for each $i \in \{0, N, 2N\}$, there are $v_1^{(i)}, v_2^{(i)}, v_3^{(i)} \in V(G)$ which are consecutive in G with $v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{w}{\mapsto} a b_{i/2N} c$. Then it cannot be the case that $v_3^{(0)} \simeq v_3^{(N)} \simeq v_3^{(2N)}$.*

Proof. Suppose for a contradiction that $f(v_3^{(0)}) = f(v_3^{(N)}) = f(v_3^{(2N)}) = k$. By Lemma 10, for each $i \in \{0, N, 2N\}$ we have either

$$v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{f}{\mapsto} (k-2)\,(k-1)\,k$$

or

$$v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{f}{\mapsto} (k+2)\,(k+1)\,k.$$

It then follows from the pidgeonhole principle that $f(v_2^{(i)}) = f(v_2^{(j)})$ for distinct $i, j \in \{0, N, 2N\}$. But this contradicts property (C3) of the chain quasi-order $\leq$. □

Lemma 13. *Suppose $v_0, \ldots, v_{|\alpha_N|-1} \in V(G)$ are consecutive in G and $v'_0, \ldots, v'_{|\beta_N|-2} \in V(G)$ are consecutive in G with $v_0 \cdots v_{|\alpha_N|-1} \overset{w}{\mapsto} \alpha_N$ and $v'_0 \cdots v'_{|\beta_N|-2} \overset{w}{\mapsto} \beta_N^-$. Suppose further that $v_0 \simeq v'_0$. Then $v_1 \not\simeq v'_1$.*

Proof. Assume without loss of generality that $v_0 \leq v_1$. Suppose for a contradiction that $v_1 \simeq v'_1$.

We know that $f(v_0) \leq f(v_1)$ and that $f(v_0) = f(v'_0)$, $f(v_1) = f(v'_1)$. Put $k := f(v_{n(0)})$, and recall that $n(0) = 6N + 5 = m(0)$. It follows from Lemma 10 that

$$v_0 \cdots v_{n(0)} \overset{f}{\mapsto} (k - 6N - 5) \cdots k, \text{ and}$$
$$v'_0 \cdots v'_{m(0)} \overset{f}{\mapsto} (k - 6N - 5) \cdots k.$$

Claim 13.1. Let $i < 2N$. If $f(v_{n(i)}) = k$ and $f(v_{n(i)+\theta(i)}) < k$, then $f(v_{n(i+1)}) = k$. Similarly, if $f(v'_{m(i)}) = k$ and $f(v'_{m(i)+\phi(i)}) < k$, then $f(v'_{m(i+1)}) = k$.

Proof of Claim 13.1. Suppose $f(v_{n(i)}) = k > f(v_{n(i)+\theta(i)})$. If

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i)),$$

then in particular $f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1$. Also, $f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 11 we must have $f(v_{n(i+1)}) = k$.

Similarly, suppose $f(v'_{m(i)}) = k > f(v'_{m(i)+\phi(i)})$. If

$$v'_{m(i)} \cdots v'_{m(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\phi(i)),$$

then in particular $f(v'_{m(i)+\phi(i)+1}) = k - \phi(i) - 1$. Also, $f(v'_{m(0)-\phi(i)-1}) = k - \phi(i) - 1$. But $w(v_{m(i)'+\phi(i)+1}) = b \not\approx_\Gamma c = w(v'_{m(0)-\phi(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 11 we must have $f(v'_{m(i+1)}) = k$. □(Claim 13.1)

Claim 13.2. Either $f(v_{n(i)}) = k$ for each $i \leq 2N$ or $f(v'_{m(i)}) = k$ for each $i \leq 2N$.

Proof of Claim 13.2. If $f(v_{n(i)+\theta(i)}) < k$ and $f(v'_{m(i)+\phi(i)}) < k$ for each $i < 2N$, then this follows immediately from Claim 13.1 and induction. Hence assume this is not the case, and let i^* be the smallest i for which $f(v_{n(i)+\theta(i)}) > k$ or $f(v'_{m(i)+\phi(i)}) > k$.

Observe that by Claim 13.1 and induction, we have $f(v_{n(i)}) = f(v'_{m(i)}) = k$ for each $i \leq i^*$.

Case 1. $f(v'_{m(i^*)+\phi(i^*)}) > k$.

It follows from Lemma 10 that

$$v'_{m(i^*)} \cdots v'_{m(i^*)+\phi(i')} \overset{f}{\mapsto} k \cdots (k + \phi(i^*)).$$

Suppose $i^* \leq i < 2N$, and that $f(v_{n(i)}) = k$. If $f(v_{n(i)+\theta(i)}) < k$, then we have by Claim 13.1 that $f(v_{n(i+1)}) = k$.

If $f(v_{n(i)+\theta(i)}) > k$, suppose for a contradiction that

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i)).$$

In particular, this means $f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1$. Also, since $\phi(i^*) > \theta(i)$, we have $f(v'_{m(i^*)+\theta(i)+1}) = k + \theta(i) + 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v'_{m(i^*)+\theta(i)+1})$, so this contradicts property (C1) of the chain quasi-order $\preceq$. Therefore by Lemma 11 we must have $f(v_{n(i+1)}) = k$.

Thus by induction, we have $f(v_{n(i)}) = k$ for each $i \leq 2N$.

Case 2. $f(v'_{m(i^*)+\phi(i^*)}) < k$ and $f(v_{n(i^*)+\theta(i^*)}) > k$.

Here we have by Claim 13.1 that $f(v'_{m(i^*+1)}) = k$.

It follows from Lemma 10 that

$$v_{n(i^*)} \cdots v_{n(i^*)+\theta(i^*)} \overset{f}{\mapsto} k \cdots (k + \theta(i^*)).$$

Suppose $i^* + 1 \leq i < 2N$, and that $f(v'_{m(i)}) = k$. If $f(v'_{m(i)+\phi(i)}) < k$, then we have by Claim 13.1 that $f(v'_{m(i+1)}) = k$.

If $f(v'_{m(i)+\phi(i)}) > k$, suppose for a contradiction that

$$v'_{m(i)} \cdots v'_{m(i+1)} \overset{f}{\mapsto} k \cdots (k + 2\phi(i)).$$

In particular, this means $f(v'_{m(i)+\phi(i)+1}) = k + \phi(i) + 1$. Also, since $\theta(i^*) > \phi(i)$, we have $f(v_{n(i^*)+\phi(i)+1}) = k + \phi(i) + 1$. But $w(v'_{m(i)+\phi(i)+1}) = b \not\approx_\Gamma c = w(v_{n(i^*)+\phi(i)+1})$, so this contradicts property (C1) of the chain quasi-order $\preceq$. Therefore by Lemma 11 we must have $f(v'_{m(i+1)}) = k$.

Thus by induction, we have $f(v'_{m(i)}) = k$ for each $i \leq 2N$. □(Claim 13.2)

It remains only to notice that Claim 13.2 contradicts Lemma 12. So we must have $v_1 \not\preceq v'_1$. □

For convenience in later statements and arguments, we will use the following notation:

Definition. Given $\sigma \in \Gamma$, define the word $\zeta_N(\sigma)$ by

$$\zeta_N(\sigma) := \begin{cases} \alpha_N & \text{if } \sigma = a \\ \beta_N & \text{if } \sigma = b \\ \gamma_N & \text{if } \sigma = c \\ \beta_N^- b_t & \text{if } \sigma = b_t \text{ (for some } t \in [0,1]). \end{cases}$$

Lemma 14. *Suppose $\sigma, \tau \in \Gamma$, $v_0, \ldots, v_\kappa \in V(G)$ are consecutive in G, and $v'_0, \ldots, v'_\lambda \in V(G)$ are consecutive in G, with*

$$v_0 \cdots v_\kappa \overset{w}{\mapsto} \zeta_N(\sigma) \quad \textit{and} \quad v'_0 \cdots v'_\lambda \overset{w}{\mapsto} \zeta_N(\tau).$$

Suppose further that $v_0 \simeq v'_0$ and $v_1 \simeq v'_1$. Then $\sigma \approx_\Gamma \tau$.

Proof. Suppose for a contradiction that $\sigma \not\approx_\Gamma \tau$. If $\sigma = a$ and $\tau \in \{b, b_t : t \in [0,1]\}$, or vice versa, then this contradicts Lemma 13. If one of them is c, say σ, then $w(v_1) = c$ while $w(v'_1) = b \not\approx_\Gamma c$, so this contradicts property (C1) of the chain quasi-order $\leq$. □

Proposition 15. *There is no chain quasi-order for ρ_N, for any N.*

Proof. Suppose for a contradiction that $\leq$ is a chain quasi-order for ρ_N. Observe that since $r, p_1, q_1 \in V(G_{\rho_N})$ are all adjacent to o in G_{ρ_N}, we have that these three vertices are also adjacent to o in the $\leq$ order. Hence by the pidgeonhole principle, some pair of them are in the relation $\simeq$. But this is a contradiction by Lemma 14. □

Oversteegen and Tymchatyn exhibit in [37] for each $\delta > 0$ a 2-dimensional plane strip with span $< \delta$ which has no chain cover of mesh < 1. Repovš et al. modify this example in [41] to construct for each $\delta > 0$ a tree in the plane with span $< \delta$ which has no chain cover of mesh < 1. In both examples, the diameters of the continua converge to ∞ as $\delta \to 0$. We pause to point out that we have now obtained a bounded family of such examples.

Corollary 16. *There is a uniformly bounded sequence $\langle T_N \rangle_{N=1}^\infty$ of simple triods in $\mathbb{R}^2$ such that for each N, $\mathrm{span}(T_N) < \frac{1}{N}$ and T_N has no chain cover of mesh $< \frac{1}{4}$.*

Proof. This is simply a combination of Propositions 5 (using T_0 and taking $\varepsilon \leq \frac{1}{2N}$), 7, 9, and 15. □

We are working to prove a stronger result: that there is a continuum in $\mathbb{R}^2$ which has span zero and cannot be covered by a chain of mesh less than some positive constant. To this end we will need some further technical combinatorial lemmas.

Lemma 17. *Suppose $\sigma, \tau \in \Gamma$ with $\sigma \approx_\Gamma \tau$, and that $v_0, \ldots, v_\kappa \in V(G)$ are consecutive in G and $v'_0, \ldots, v'_\kappa \in V(G)$ are consecutive in G with*

$$v_0 \cdots v_\kappa \overset{w}{\mapsto} \zeta_N(\sigma) \quad \textit{and} \quad v'_0 \cdots v'_\kappa \overset{w}{\mapsto} \zeta_N(\tau).$$

Then:

(i) if $v_0 < v_1$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;

(ii) if $v_{\kappa-1} < v_\kappa$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;

(iii) if $v_0 \simeq v'_0$ and $v_1 \simeq v'_1$, then $v_\kappa \simeq v'_\kappa$; and

(iv) if $v_\kappa \simeq v'_\kappa$ and $v_{\kappa-1} \simeq v'_{\kappa-1}$, then $v_0 \simeq v'_0$.

Proof. Each of these statements is trivial if $\sigma = \tau = c$. We will prove the Lemma for $\sigma = \tau = a$; the case $\sigma \approx_\Gamma \tau \approx_\Gamma b$ proceeds analogously.

(i) Suppose $v_0 < v_1$.

Claim 17.1. $v_0 \cdots v_{n(0)} \overset{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)})$.

Proof of Claim 17.1. This is immediate from Lemma 10. □(Claim 17.1)

Claim 17.2. For each $i < 2N$, $v_{n(i)} \leq v_{n(i+1)}$.

Proof of Claim 17.2. We proceed by induction on $i < 2N$. Suppose the claim is true for each i' with $i' < i$. Put $k := f(v_{n(i)})$. Suppose for a contradiction that $f(v_{n(i)}) > f(v_{n(i+1)})$. By Lemma 11, this means

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i)).$$

In particular, we have $f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1$.

Let j^* be the smallest $j \leq i$ such that $f(v_{n(j)}) = k$.

If $j^* = 0$, then since $n(0) > \theta(i)$, we have $f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1$. But also $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$.

If $j^* > 0$, then we know by Lemma 11 that

$$v_{n(j^*-1)} \cdots v_{n(j^*)} \overset{f}{\mapsto} (k - 2\theta(j^* - 1)) \cdots k.$$

Then similarly observe that since $\theta(j^* - 1) > \theta(i)$, we have $f(v_{n(j^*)-\theta(i)-1}) = k - \theta(i) - 1$. But also $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(j^*)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. □(Claim 17.2)

Claim 17.3. $v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5)$.

Proof of Claim 17.3. By Lemma 12 and Claim 17.2, we must have $v_{n(i-1)} < v_{n(i)}$ for some $0 < i \leq 2N$; let i^* be the largest such i, so that $f(v_{n(2N)}) = f(v_{n(i^*)})$.

Suppose for a contradiction that

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) - 6N - 5).$$

Then in particular, since $6N+5 > \theta(i^*-1)$, we have $f(v_{n(2N)+\theta(i^*-1)+1}) = f(v_{n(2N)}) - \theta(i^*-1)-1$. But also $f(v_{n(i^*)-\theta(i^*-1)-1}) = f(v_{n(2N)})-\theta(i^*-1)-1$ and $w(v_{n(i^*)-\theta(i^*-1)-1}) = c \not\approx_\Gamma a = w(v_{n(2N)+\theta(i^*-1)+1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 10, we must have

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5).$$

□(Claim 17.3)

It is now easy to check that $f(v_0) = f(v_{n(0)}) - 6N - 5 < f(v_j) < f(v_{n(2N)}) + 6N + 5 = f(v_\kappa)$ for any $0 < j < \kappa$.

(ii) Observe that if we consider the reverse order of $\leq$, part (i) gives that if $v_0 > v_1$, then $v_0 > v_j > v_\kappa$ for each $0 < j < \kappa$. In particular, this would mean $v_{\kappa-1} > v_\kappa$. Therefore if $v_{\kappa-1} < v_\kappa$ then $v_0 < v_1$, hence the conclusion follows from part (i).

(iii) Suppose $v_0 \simeq v'_0$, $v_1 \simeq v'_1$, and assume without loss of generality that $v_0 < v_1$. This means Claims 17.1, 17.2, and 17.3 hold for the v_j's and the v'_j's. By Claim 17.1, we have

$$v_0 \cdots v_{n(0)} \overset{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)})$$

and

$$v'_0 \cdots v'_{n(0)} \overset{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)}).$$

Claim 17.4. For each $i \leq 2N$, $v_{n(i)} \simeq v'_{n(i)}$.

Proof of Claim 17.4. Suppose not, and let i^* be the smallest $i < 2N$ such that $v_{n(i+1)} \not\simeq v'_{n(i+1)}$. Put $k := f(v_{n(i^*)}) = f(v'_{n(i^*)})$. It follows from Lemma 11 and Claim 17.2 that either $f(v_{n(i^*+1)}) = k$ and $f(v'_{n(i^*+1)}) > k$, or $f(v_{n(i^*+1)}) > k$ and $f(v'_{n(i^*+1)}) = k$; assume the former. This implies by Lemma 11 that

$$v'_{n(i^*)} \cdots v'_{n(i^*+1)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i^*)).$$

We claim that $f(v_{n(i)}) = k$ for each $i \geq i^*$. Indeed, given $i > i^*$, suppose for a contradiction that

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i)).$$

This means in particular that $f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1$. Since $\theta(i) < \theta(i^*)$, we have $f(v'_{n(i^*)+\theta(i)+1}) = k + \theta(i) + 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v'_{n(i^*)+\theta(i)+1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 11 and Claim 17.2, we must have $f(v_{n(i+1)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \geq i^*$.

In particular, $f(v_{n(2N)}) = k$. By Claim 17.3, we have

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} k \cdots (k + 6N + 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(2N)+\theta(i^*)+1}) = k + \theta(i^*) + 1$. Note $f(v'_{n(i^*)+\theta(i^*)+1}) = k+\theta(i^*)+1$ as well. But $w(v'_{n(i^*)+\theta(i^*)+1}) = c \not\approx_\Gamma a = w(v_{n(2N)+\theta(i^*)+1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. □(Claim 17.4)

Claim 17.4 implies in particular that $f(v_{n(2N)}) = f(v'_{n(2N)})$. Then by Claim 17.3, we have

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5),$$

and

$$v'_{n(2N)} \cdots v'_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5).$$

This establishes part (iii).

(iv) Suppose $v_\kappa \simeq v'_\kappa$, $v_{\kappa-1} \simeq v'_{\kappa-1}$, and assume without loss of generality that $v_{\kappa-1} < v_\kappa$. By part (ii) this implies $v_0 < v_1$ and $v'_0 < v'_1$, so again Claims 17.1, 17.2, and 17.3 hold for the v_j's and the v'_j's. By Claim 17.3, we have

$$v_\kappa \cdots v_{n(2N)} \overset{f}{\mapsto} (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)})$$

and

$$v'_\kappa \cdots v'_{n(2N)} \overset{f}{\mapsto} (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)}).$$

Claim 17.5. For each $i \leq 2N$, $v_{n(i)} \simeq v'_{n(i)}$.

Proof of Claim 17.5. Suppose not, and let i^* be the largest $i < 2N$ such that $v_{n(i)} \not\simeq v'_{n(i)}$. Put $k := f(v_{n(i^*+1)}) = f(v'_{n(i^*+1)})$. It follows from Lemma 11 and Claim 17.2 that either $f(v_{n(i^*)}) = k$ and $f(v'_{n(i^*)}) < k$, or $f(v_{n(i^*)}) < k$ and $f(v'_{n(i^*)}) = k$; assume the former. This implies by Lemma 11 that

$$v'_{n(i^*+1)} \cdots v'_{n(i^*)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i^*)).$$

We claim that $f(v_{n(i)}) = k$ for each $i \leq i^*$. Indeed, given $i < i^*$, suppose for a contradiction that

$$v_{n(i+1)} \cdots v_{n(i)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i)).$$

Since $\theta(i^*) < \theta(i)$, this means in particular that $f(v_{n(i+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$. Note $f(v'_{n(i^*+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$ as well. But $w(v_{n(i^*+1)-\theta(i^*)-1}) = c \not\approx_\Gamma a = w(v'_{n(i+1)-\theta(i^*)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 11 and Claim 17.2 we must have $f(v_{n(i)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \leq i^*$.

In particular, $f(v_{n(0)}) = k$. By Claim 17.1, we have

$$v_{n(0)} \cdots v_0 \overset{f}{\mapsto} k \cdots (k - 6N - 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(0)-\theta(i^*)-1}) = k - \theta(i^*) - 1$. Note $f(v'_{n(i^*+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$ as well. But $w(v'_{n(i^*+1)-\theta(i^*)-1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i^*)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. □(Claim 17.5)

Claim 17.5 implies in particular that $f(v_{n(0)}) = f(v'_{n(0)})$. Then by Claim 17.1, we have

$$v_{n(0)} \cdots v_0 \overset{f}{\mapsto} f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5)$$

and

$$v'_{n(0)} \cdots v'_0 \overset{f}{\mapsto} f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5).$$

This establishes part (iv).

□

3.4.2 Iterated sketches

If $\iota_T : \Gamma \to T$ is a Γ-marking of the simple triod T and ρ_N is a $\langle T, \varepsilon\rangle$-sketch of the simple triod graph $T' := G_{\rho_N}$ such that $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$ (as in Proposition 5), then one can define an induced Γ-marking $\iota_{T'} : \Gamma \to T'$ on T' as follows: define $\iota_{T'}(a) := p_{|\alpha_N|-1}$, $\iota_{T'}(b) := q_{|\beta_N|-1} = \iota_T(b)$, $\iota_{T'}(c) := r$, and for each $t \in [0,1]$ put $\iota_{T'}(b_t) := \iota_T(b_t) \in [q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$.

Now let T_0 be as before, and suppose T_1 and T_2 are simple triods such that ρ_1 is a $\langle T_0, \varepsilon_0\rangle$-sketch of T_1, and ρ_2 is a $\langle T_1, \varepsilon_1\rangle$-sketch of T_2 (using the induced Γ-marking on T_1). Evidently we should be able to find a $\langle T_0, \varepsilon_0 + \varepsilon_1\rangle$-sketch of T_2, and indeed this is necessary if we want to apply Proposition 9 to argue that T_2 has no chain cover of small mesh. This is the motivation for the next definition (see Proposition 18).

Definition. Suppose $\langle G, w\rangle$ is a compliant graph-word, and $N > 0$. A graph-word $\langle G^+, w^+\rangle$ is a *ρ_N-expansion of* $\langle G, w\rangle$ if:

- G^+ is identical to G as a topological space, but the vertex set of G^+ is finer: for any adjacent pair of vertices $v_1, v_2 \in V(G)$, there are distinct degree 2 vertices $s_j^{v_1v_2}$, $j = 1, \ldots, \kappa_{v_1v_2}$ where $\kappa_{v_1v_2} = |\zeta_N(w(v_1))| + |\zeta_N(w(v_2))| - 3$, inserted into the edge joining v_1, v_2 so that $v_1, s_1^{v_1v_2}, \ldots, s_{\kappa_{v_1v_2}}^{v_1v_2}, v_2$ are consecutive in G^+; and
- w^+ is defined by

$$v_1 s_1^{v_1v_2} \cdots s_{\kappa_{v_1v_2}}^{v_1v_2} v_2 \overset{w^+}{\mapsto} \zeta_N(w(v_1))^{\leftarrow} \pitchfork \zeta_N(w(v_2))$$

  when $v_1, v_2 \in V(G)$ are adjacent in G.

Remarks. (1) Notice that $w^+|_{V(G)} = w$, and that $\langle G^+, w^+\rangle$ is also a compliant graph-word.

(2) Combinatorially, there is only one ρ_N expansion of a given graph-word $\langle G, w\rangle$; however, geometrically they may differ according to where along the edges of G the extra vertices are inserted (though their order on the edge is determined uniquely by the definition).

Proposition 18. *Let T be a Γ-marked simple triod. Suppose ρ_N is a $\langle T, \varepsilon_1\rangle$-sketch of $T' := G_{\rho_N}$, and that $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T'} = [\iota_T(c), \iota_T(b)]_T$ (as in Proposition 5). Endow T' with a Γ-marking as in the beginning of subsection 3.4.2. If $\rho = \langle G, w\rangle$ is a compliant graph-word which is a $\langle T', \varepsilon_2\rangle$-sketch of G, then there is a ρ_N-expansion of $\langle G, w\rangle$ which is a $\langle T, \varepsilon_1 + \varepsilon_2\rangle$-sketch of G.*

Proof. Let $\widehat{w_{\rho_N}} : T' \to T$ be a ρ_N-suggested bonding map such that $d(x, \widehat{w_{\rho_N}}(x)) < \frac{\varepsilon_1}{2}$ for each $x \in T'$, and let $\widehat{w} : G \to T'$ be a ρ-suggested bonding map such that $d(x, \widehat{w}(x)) < \frac{\varepsilon_2}{2}$ for each $x \in G$.

Consider any adjacent $v_1, v_2 \in V(G)$. Define

$$s_i^{v_1 v_2} := \begin{cases} (\widehat{w}|_{[v_1,v_2]})^{-1}(p_{|\alpha_N|-1-i}) & \text{if } w(v_1) = a \\ (\widehat{w}|_{[v_1,v_2]})^{-1}(q_{|\beta_N|-1-i}) & \text{if } w(v_1) \approx_\Gamma b \\ (\widehat{w}|_{[v_1,v_2]})^{-1}(o) & \text{if } w(v_1) = c \end{cases}$$

for $1 \le i \le |\zeta_N(w(v_1))| - 1$, and

$$s_{\kappa_{v_1 v_2}+1-i}^{v_1 v_2} := \begin{cases} (\widehat{w}|_{[v_1,v_2]})^{-1}(p_{|\alpha_N|-1-i}) & \text{if } w(v_2) = a \\ (\widehat{w}|_{[v_1,v_2]})^{-1}(q_{|\beta_N|-1-i}) & \text{if } w(v_2) \approx_\Gamma b \end{cases}$$

for $1 \le i \le |\zeta_N(w(v_2))| - 2$.

Let $V(G^+)$ be equal to $V(G)$ together with all these new vertices, and let w^+ be defined as in the definition of a ρ_N-expansion. Observe that $w^+ = w_{\rho_N} \circ (\widehat{w}|_{V(G^+)})$. Put $\rho^+ := \langle G^+, w^+ \rangle$, where G^+ is equal to G as a topological space, with vertex set $V(G^+)$.

It is now straightforward to see that $\widehat{w_{\rho_N}} \circ \widehat{w}$ is a ρ^+-suggested bonding map, and clearly $d(x, (\widehat{w_{\rho_N}} \circ \widehat{w})(x)) < \frac{\varepsilon_1 + \varepsilon_2}{2}$ for each $x \in G$. □

Lemma 19. *Suppose $\langle G, w \rangle$ is a compliant graph-word, let $\langle G^+, w^+ \rangle$ be a ρ_N-expansion of $\langle G, w \rangle$, and suppose $\le^+$ is a chain quasi-order of $\langle G^+, w^+ \rangle$.*

(i) Let $v_1, v_2 \in V(G)$ be adjacent in G, and let $s_1, \ldots, s_\kappa \in V(G^+) \setminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in G^+. Then the following are equivalent:

(1) $v_1 <^+ v_2$;

(2) $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$;

(3) $v_1 <^+ s_j <^+ v_2$ for some $j \in \{1, \ldots, \kappa\}$.

(ii) If $v_1, v_2 \in V(G)$ are adjacent in G and $v_1', v_2' \in V(G)$ are adjacent in G with $v_1 \simeq^+ v_1'$, $v_1 <^+ v_2$, and $v_1' <^+ v_2'$, then $v_2 \simeq^+ v_2'$.

Proof. (i) The implications (2) ⇒ (3) and (3) ⇒ (1) are trivial. For (1) ⇒ (2) we will prove that $v_1 <^+ s_1$ implies that $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$. Then by considering the reverse order of $\leq^+$, it follows that $v_1 <^+ v_2$ implies $v_1 <^+ s_1$, hence $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$.

Suppose $v_1 <^+ s_1$. Let $i \in \{1, \ldots, \kappa\}$ be such that

$$s_i \cdots s_1 v_1 \overset{w^+}{\mapsto} \zeta_N(w(v_1)) \quad \text{and} \quad s_i \cdots s_\kappa v_2 \overset{w^+}{\mapsto} \zeta_N(w(v_2)).$$

By Lemma 17 (ii), we have $v_1 <^+ s_j <^+ s_i$ for each $j \in \{1, \ldots, i-1\}$. Because G is compliant, we can deduce using Lemma 14 that $s_i <^+ s_{i+1}$. Then by Lemma 17 (i) we have $s_i <^+ s_j <^+ v_2$ for each $j \in \{i+1, \ldots, \kappa\}$.

(ii) Suppose $v_1, v_2 \in V(G)$ are adjacent in G and $v_1', v_2' \in V(G)$ are adjacent in G with $v_1 \simeq^+ v_1'$, $v_1 <^+ v_2$, and $v_1' <^+ v_2'$. Let $s_1, \ldots, s_\kappa$ and i be as in part (i), and let $s_1', \ldots, s_\lambda' \in V(G^+) \setminus V(G)$ be such that $v_1' s_1', \ldots, s_\lambda', v_2'$ are consecutive in G^+ and

$$v_1' s_1' \cdots s_\lambda' v_2' \overset{w^+}{\mapsto} \zeta_N(w(v_1'))^{\leftarrow} \pitchfork \zeta_N(w(v_2')).$$

By property (C1) of the chain quasi-order $\leq^+$, $w(v_1) \approx_\Gamma w(v_1')$, hence $|\zeta_N(v_1)| = |\zeta_N(v_1')|$, and so

$$s_i' \cdots s_1' v_1' \overset{w^+}{\mapsto} \zeta_N(w(v_1')) \quad \text{and} \quad s_i' \cdots s_\lambda' v_2' \overset{w^+}{\mapsto} \zeta_N(w(v_2')).$$

By Lemma 17 (iv), we have $s_i \simeq^+ s_i'$, and as in part (i) we know that $s_{i+1}' >^+ s_i'$. By Lemma 14, this implies $w(v_2) \approx_\Gamma w(v_2')$, hence $\kappa = \lambda$. Then by Lemma 17 (iii), we conclude that $v_2 \simeq^+ v_2'$.

□

Proposition 20. *Suppose $\langle G, w\rangle$ is a compliant graph-word. If a (any) ρ_N-expansion of $\langle G, w\rangle$ has a chain quasi-order, then $\langle G, w\rangle$ also has a chain quasi-order.*

Proof. Let $\langle G^+, w^+\rangle$ be a ρ_N-expansion of $\langle G, w\rangle$, and let $\leq^+$ be a chain quasi-order of $\langle G^+, w^+\rangle$.

Define $\leq$ on $V(G)$ by $\leq \, := \, \leq^+|_{V(G)}$. Clearly $\leq$ is a total quasi-order since $\leq^+$ is. We must check that $\leq$ satisfies properties (C1), (C2), and (C3) of the definition of a chain quasi-order.

(C1): This is immediate since $\leq^+$ satisfies this property.

(C2): We will need the following claim:

Claim 20.1. In $\langle G^+, w^+\rangle$, if $v \in V(G)$ and $v' \in V(G^+)$ are such that $v \simeq^+ v'$, then in fact $v' \in V(G)$.

Proof of Claim 20.1. We proceed by induction on the number of vertices in G.

If $|V(G)| = 1$, then there is nothing to prove.

Assume the claim holds for all such graph-words whose graph has n or fewer vertices, and assume $|V(G)| = n + 1$. Let $u \in V(G)$ be such that the subgraph G^- obtained by removing the vertex u (and all edges emanating from u) is connected. There is a ρ_N-expansion of $\langle G^-, w|_{V(G)\smallsetminus\{u\}}\rangle$ which is a sub-graph-word of $\langle G^+, w^+\rangle$ (it has vertex set $V(G^+)\cap G^-$), and the restriction of $\leq^+$ to this sub-graph-word is a chain quasi-order. By induction, the claim holds for G^-.

Let $u' \in V(G) \smallsetminus \{u\}$ be adjacent to u in G. Let $s_1 \dots, s_\kappa \in V(G^+) \smallsetminus V(G)$ be such that $u', s_1, \dots, s_\kappa, u$ are consecutive in G^+ and

$$u' s_1 \cdots s_\kappa u \overset{w^+}{\mapsto} \zeta_N(w(u'))^{\leftarrow} \pitchfork \zeta_N(w(u)).$$

Assume $u' <^+ u$ (the other case proceeds similarly), which implies by Lemma 19 (i) that $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$.

We have four things to check:

(1) for each $y \in V(G) \smallsetminus \{u\}$ and each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $y \not\simeq^+ s$;

(2) for each $y \in V(G) \smallsetminus \{u\}$ and each $j \in \{1, \dots, \kappa\}$, $y \not\simeq^+ s_j$;

(3) for each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $u \not\simeq^+ s$; and

(4) for each $j \in \{1, \dots, \kappa\}$, $u \not\simeq^+ s_j$.

Observe that (1) holds by induction, and (4) is immediate from the observation that $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$. For (2) and (3), we consider two cases.

Case 1. For every $y \in V(G) \smallsetminus \{u\}$, $y \leq^+ u'$.

Since $u' <^+ s_j <^+ u$ for each $j \in \{1, \dots, \kappa\}$, we have immediately that $y \not\simeq^+ s_j$ for any $y \in V(G) \smallsetminus \{u\}$.

Also, from Lemma 19 (i) it follows that for every $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $s <^+ u'$. Therefore $u \not\simeq^+ s$ for any such s.

Case 2. There exists some $y \in V(G) \smallsetminus \{u\}$ such that $u' <^+ y$.

Let $\mathcal{P}$ be a path of vertices in G^- starting at u' and ending at y. Let y_1 be the latest vertex y' in $\mathcal{P}$ with $y' \leq^+ u'$, and let y_2 be the next vertex in $\mathcal{P}$ after y_1, so that y_1 and y_2 are adjacent in G and $y_1 \leq^+ u' <^+ y_2$.

Suppose for a contradiction that $y_1 <^+ u'$. Let $z_1, \ldots, z_\lambda \in V(G^+) \smallsetminus V(G)$ be such that $y_1, z_1, \ldots, z_\lambda, y_2$ are consecutive in G^+. Then by Lemma 8 there is some $i \in \{1, \ldots, \lambda\}$ such that $u' \simeq^+ z_i$. But the claim holds for G^- by induction, so this is a contradiction. Therefore we must have $u' \simeq^+ y_1$.

Then from Lemma 19 (ii) we know that $u \simeq^+ y_2$. It follows immediately that $u \not\simeq^+ s$ for each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, because $y_2 \not\simeq^+ s$ for every such s by induction.

Moreover, for each $j \in \{1, \ldots, \kappa\}$, since $y_1 \simeq^+ u' <^+ s_j <^+ u \simeq^+ y_2$, we know from Lemma 8 that there is some $s \in V(G^+) \smallsetminus V(G)$ inserted between y_1 and y_2 such that $s_j \simeq^+ s$. It follows that $y \not\simeq^+ s_j$ for any $y \in V(G) \smallsetminus \{u\}$, because $y \not\simeq^+ s$ for every such y by induction. $\square$(Claim 20.1)

Now suppose $v_1, v_2 \in V(G)$ are adjacent in G, and assume $v_1 \leq v_2$. Let $s_1, \ldots, s_\kappa \in V(G^+) \smallsetminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in $V(G^+)$. If $v \in V(G)$ were such that $v_1 < v < v_2$, then $v_1 <^+ v <^+ v_2$ as well, so by Lemma 8 there would be some $i \in \{1, \ldots, \kappa\}$ such that $v \simeq^+ s_i$. But this contradicts Claim 20.1.

(C3): Suppose $v \in V(G)$, v_1, v_2, v_3 are consecutive in G, and that $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0, 1]$ are such that $t' \geq t$, $w(v) = b_{t'}$, $v_1 v_2 v_3 \overset{w}{\mapsto} \sigma b_t \tau$, and $v_1 < v_2 \simeq v < v_3$.

Let $s_1, \ldots, s_\kappa, s'_1, \ldots, s'_\lambda \in V(G^+) \smallsetminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2, s'_\lambda, \ldots, s'_1, v_3$ are consecutive in G^+, and

$$v_1 s_1 \cdots s_\kappa v_2 s'_\lambda \cdots s'_1 v_3 \overset{w^+}{\mapsto} \zeta_N(\sigma)^{\leftarrow} \pitchfork \beta_N^- b_t (\beta_N^-)^{\leftarrow} \pitchfork \zeta_N(\tau).$$

Observe that $w^+(s_\kappa) = w^+(s'_\lambda) = c$. Since $v_1 <^+ v_2$, by Lemma 19 (i) we must have $s_\kappa <^+ v_2$. Likewise, we have $v_2 <^+ s'_\lambda$. It now follows from property (C3) of the chain quasi-order $\leq^+$ that $t' - t < \frac{1}{2}$. $\square$

3.5 The example

Example 1. There exists a continuum $X \subset \mathbb{R}^2$ which is non-chainable and has span zero.

Proof. First we define by recursion a sequence $\langle T_N\rangle_{N=0}^{\infty}$ of Γ-marked simple triods in $\mathbb{R}^2$ and a sequence $\langle \varepsilon_N\rangle_{N=0}^{\infty}$ of positive real numbers as follows.

Let $T_0 \subset \mathbb{R}^2$ be as defined in subsection 3.2.1, and put $\varepsilon_0 := \frac{1}{8}$.

Suppose T_N, ε_N have been defined. Apply Proposition 5 to obtain an embedding T_{N+1} of the simple triod graph $G_{\rho_{N+1}}$ in $\mathbb{R}^2$ such that ρ_{N+1} is a $\langle T_N, \varepsilon_N\rangle$-sketch of T_{N+1}, and $[q_{|\beta_N|-2}, q_{|\beta_N|-1}]_{T_{N+1}} = [\iota_{T_N}(c), \iota_{T_N}(b)]_{T_N}$. Endow T_{N+1} with a Γ-marking as in the beginning of subsection 3.4.2. Notice that $T_{N+1} \subset (T_N)_{\varepsilon_N}$, where Y_ε denotes the ε-neighborhood of the space Y in $\mathbb{R}^2$. By Proposition 7, the span of T_{N+1} is $< \frac{1}{2(N+1)} + \varepsilon_N$. Let $0 < \varepsilon_{N+1} < 2^{-N-4}$ be small enough so that $\overline{(T_{N+1})_{\varepsilon_{N+1}}} \subseteq \overline{(T_N)_{\varepsilon_N}}$, and so that $\mathrm{span}(\overline{(T_{N+1})_{\varepsilon_{N+1}}}) < \frac{1}{2(N+1)} + 2\varepsilon_N$.

Put $X := \bigcap_{N=0}^{\infty} \overline{(T_N)_{\varepsilon_N}}$.

Observe that for any N, we have $X \subseteq \overline{(T_{N+1})_{\varepsilon_{N+1}}}$, hence

$$\mathrm{span}(X) \leq \mathrm{span}(\overline{(T_{N+1})_{\varepsilon_{N+1}}}) < \frac{1}{2(N+1)} + 2\varepsilon_N.$$

Since ε_N converges to 0 as $N \to \infty$, it follows that X has span zero.

Suppose for a contradiction that X has a chain cover of mesh $< \frac{1}{4}$. Then there is some $N > 0$ for which T_N has a chain cover of mesh $< \frac{1}{4}$.

Define by recursion the graph-words $\langle G_i, w_i\rangle$, $0 \leq i \leq N-1$, as follows: $\langle G_{N-1}, w_{N-1}\rangle := \rho_N$, and for $i < N-1$, $\langle G_i, w_i\rangle$ is the ρ_{i+1}-expansion of $\langle G_{i+1}, w_{i+1}\rangle$ provided by Proposition 18 which is a $\langle T_i, \sum_{j=i}^{N-1} \varepsilon_j\rangle$-sketch of T_N. In particular, $\langle G_0, w_0\rangle$ is a $\langle T_0, \sum_{j=0}^{N-1} \varepsilon_j\rangle$-sketch of T_N.

Since $\sum_{j=0}^{N-1} \varepsilon_j < \sum_{j=0}^{N-1} 2^{-j-3} < \frac{1}{4}$, by Proposition 9 we have that $\langle G_0, w_0\rangle$ has a chain quasi-order. Then by Proposition 20 and induction, we obtain a chain quasi-order for each graph-word $\langle G_i, w_i\rangle$. In particular, $\langle G_{N-1}, w_{N-1}\rangle$ has a chain quasi-order. But $\langle G_{N-1}, w_{N-1}\rangle$ is ρ_N, so this contradicts Proposition 15. □

3.6 Questions

The construction presented in this chapter can be carried out so that every proper subcontinuum of the resulting space is an arc; hence, in particular, it is far from being hereditarily indecomposable. On the other hand, as discussed in the Introduction, if there exists a non-degenerate homogeneous continuum in the plane which is not homeomorphic to the circle, the pseudo-arc, or the circle of pseudo-arcs, then there would be one which is hereditarily indecomposable and with span zero. Given that the pseudo-arc is the only hereditarily indecomposable chainable continuum [4], this raises the following question:

Question 2 (See Problem 9 of [39])**.** *Is there a hereditarily indecomposable non-chainable continuum with span zero?*

If such an example exists, then by [39, Corollary 6] it would be a continuous image of the pseudo-arc. Since any map to a hereditarily indecomposable continuum is confluent [42, Lemma 15], it would also be a counterexample to Problem 84 of [8], which asks whether every confluent image of a chainable continuum is chainable.

Regarding the planarity of the example in this chapter, while every chainable continuum can be embedded in the plane [4], the same is not known to be true of continua with span zero.

Question 3. *Can every continuum with span zero be embedded in* $\mathbb{R}^2$*?*

Chapter 4

Copies in the plane

Recall from the discussion in the Introduction that it if Y is a non-degenerate, non-separating, homogeneous plane continuum which is not homeomorphic to the pseudo-arc, then Y must be tree-like (in fact span-zero), non-chainable, and hereditarily indecomposable. Moreover, Y must contain uncountably many pairwise disjoint copies of some proper non-chainable subcontinuum X.

Up to this point, there were no known examples of a non-chainable tree-like continuum X having the property that there is an uncountable family of pairwise disjoint homeomorphic copies of X in the plane $\mathbb{R}^2$. Ingram [14] has constructed an uncountable family of pairwise disjoint non-chainable tree-like continua in the plane, but these continua are not pairwise homeomorphic (in fact, no single continuum maps onto every member of the family).

It is the purpose of this chapter to show that the continuum X from Chapter 3 has the property that $X \times \mathcal{C}$ embeds in the plane, where $\mathcal{C}$ is the middle-thirds Cantor set. By a result of Todorčević [46], this is equivalent to the seemingly weaker statement that there is an uncountable family of pairwise disjoint homeomorphic copies of X in the plane.

4.1 Preliminaries

The proof in the next section will depend heavily upon the definitions and notation introduced in Chapter 3. We will again use *graph-words* in the alphabet $\Gamma = \{a, b, c, b_t : t \in [0, 1]\}$, particularly the graph-word ρ_N, discussed in Section 3.2. We will also make reference to the concepts of a Γ-*marking* of a simple triod, a ρ_N-*suggested bonding map*, and a *sketch*, also defined in Section 3.2.

Denote by SEQ the set of finite sequences of zeros and ones, including the empty sequence $\emptyset$. For $s \in \mathsf{SEQ}$, $|s|$ denotes the length of s, and if $j \in \{0,1\}$, then $s \frown j$ is the sequence obtained by adding a single j to the end of s.

We will identify the Cantor set $\mathcal{C}$ with the set of functions from ω to $\{0,1\}$. Given $y \in \mathcal{C}$ and $n \in \omega$, denote by $y \restriction n$ the restriction of y to n, which is an element of SEQ of length n.

For convenience in the proof below, we will make use of the following result of Todorčević:

Theorem (Todorčević, [46]). *If Y is a separable complete metric space and X is a compact metric space, then either the set of homeomorphic copies of X in Y is σ-linked (in particular does not contain any uncountable family of pairwise disjoint elements), or $X \times \mathcal{C}$ can be embedded in Y.*

4.2 Uncountably many copies of X in the plane

Example 2. There is a tree-like, non-chainable continuum X such that $X \times \mathcal{C}$ embeds in the plane $\mathbb{R}^2$, where $\mathcal{C}$ is the middle-thirds Cantor set.

Proof. For convenience, let $G_{\rho_0} := T_0$, with Γ-marking $\iota_0 : \Gamma \to G_{\rho_0}$ as defined in Section 3.2. Recursively, for each $N \in \omega$, fix a ρ_N-suggested bonding map $f_N : G_{\rho_{N+1}} \to G_{\rho_N}$, and define a Γ-marking ι_{N+1} of $G_{\rho_{N+1}}$ as follows: let $\iota_{N+1}(\sigma)$ equal the endpoint of $G_{\rho_{N+1}}$ which is mapped to $\iota_N(\sigma)$ by f_N, for $\sigma \in \{a, b, c\}$, and for $t \in [0,1]$ let $\iota_{N+1}(b_t)$ equal the point p that is in the component of $\iota_{N+1}(b)$ in the set $f_N^{-1}([\iota_N(b_0), \iota_N(b_1)])$, such that $f_N(p) = \iota_N(b_t)$).

We will carry out the construction from Chapter 3 once for each point of $\mathcal{C}$, to obtain uncountably many non-chainable tree-like continua in the plane. We will take care to ensure these continua are pairwise disjoint, and that each is homeomorphic to the inverse limit X of the triod graphs G_{ρ_N} with the bonding maps f_N. Once this is done, it follows immediately from the result of Todorčević quoted above that $X \times \mathcal{C}$ embeds in the plane.

To begin, we observe that if T is a Γ-marked simple triod in $\mathbb{R}^2$, $\varepsilon > 0$, and N is a positive integer, then we can find two disjoint embeddings Ψ_1, Ψ_2 of the simple triod graph G_{ρ_N} into $(T)_\varepsilon$, the ε-neighborhood of T, such that ρ_N is a $\langle T, \varepsilon + \frac{1}{N} \rangle$-sketch of both $\Psi_1(G_{\rho_N})$ and $\Psi_2(G_{\rho_N})$. This can be seen by looking at the proof of Proposition 5 from Section 3.2 of Chapter 3: we note that in the first intermediate embedding stage (refer to Figure 3.1), we can embed two disjoint copies of G_{ρ_N} as depicted in Figure 4.1. Then, after carrying out the two subsequent transformations described in the proof of Proposition 5 (seen in Figures 3.2 and 3.3), we see that all points of both embeddings of G_{ρ_N} can be within $\frac{\varepsilon}{2}$ of their images in T, except for those points near vertices mapped into $\{b_t : t \in [0,1)\}$ by ρ_N – these will be within $\frac{\varepsilon}{2} + \frac{1}{2N}$ of their images in T.

Now let $T_\emptyset = T_0$. Given a sequence $\langle \varepsilon_N \rangle_{N\in\omega}$ of positive real numbers whose infinite series converges, we can use the above observation to recursively construct embeddings Ψ_s of $G_{\rho_{|s|}}$ into $\mathbb{R}^2$, for $s \in \mathsf{SEQ}$, and denote $T_s := \Psi_s(G_{\rho_{|s|}})$, such that for each $s \in \mathsf{SEQ}$ and $j \in \{0,1\}$, $T_{s^\frown j} \subset (T_s)_{\varepsilon_{|s|}}$, $\rho_{|s|+1}$ is a $\langle T_s, \varepsilon_{|s|} + \frac{1}{|s|}\rangle$-sketch of $T_{s^\frown j}$, and $T_{s^\frown 0} \cap T_{s^\frown 1} = \emptyset$. Moreover, by taking the sequence of ε_N's converging to zero fast enough, we may assume that $\overline{(T_{s^\frown j})_{\varepsilon_{|s|+1}}} \subset (T_s)_{\varepsilon_{|s|}}$ for each s and j. We can then define, for $y \in \mathcal{C}$, the continuum $X_y := \bigcap_{N\in\omega} \overline{(T_{y\restriction N})_{\varepsilon_N}}$. Observe that $X_y = \bigcap_{N\in\omega} \overline{\bigcup_{k\geq N} T_{y\restriction k}}$.

By taking the ε_N's converging to zero fast enough, we may also assume that the closures of the ε_N-neighborhoods of T_s and $T_{s'}$ are disjoint whenever $s \neq s'$ and $|s| = |s'| = N$. This ensures that if $y, y' \in \mathcal{C}$ with $y \neq y'$, then $X_y \cap X_{y'} = \emptyset$.

To ensure that all the continua X_y are pairwise homeomorphic, we need to impose some additional restrictions on the embeddings Ψ_s. We begin with some notation. Fix $y \in \mathcal{C}$.

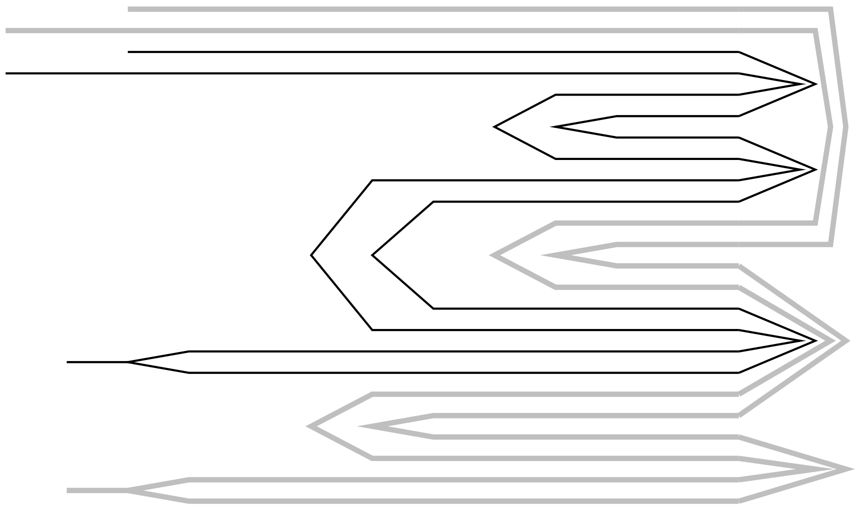
Denote $T_N := T_{y\restriction N}$ and define $g_N : T_{N+1} \to T_N$ by $g_N := \Psi_{y\restriction N} \circ f_N \circ \Psi^{-1}_{y\restriction(N+1)}$ for each $N \in \omega$. Observe that g_N is a ρ_{N+1}-suggested bonding map for each $N \in \omega$. If $k < N$, define $g_k^N : T_N \to T_k$ by $g_k^N := g_k \circ g_{k+1} \circ \cdots \circ g_{N-1}$.

Let us augment the vertex set of each T_N to obtain T_N^+, by setting $T_1^+ = T_1$, and then recursively adding the preimages of every vertex of T_N^+ under the map g_N to the vertex set of T_{N+1} to obtain T_{N+1}^+. Since for any adjacent vertices $v, v' \in T_N$ the restriction of g_{N-1} to the arc $[v, v']$ is a homeomorphism (by definition of a ρ_N-suggested bonding map), we immediately have:

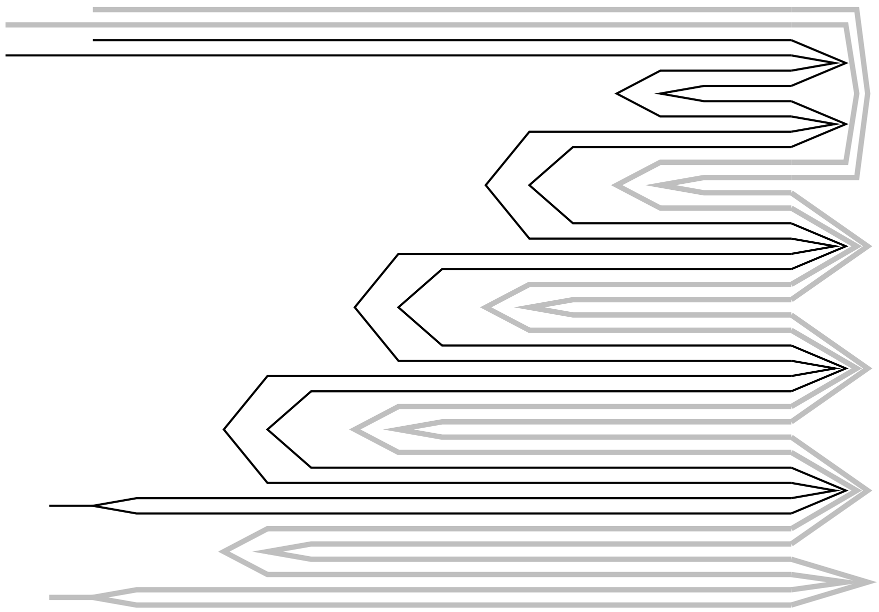
(1) If v, v' are adjacent vertices in T_N^+, then g_{N-1} maps $[v, v']$ homeomorphically onto $[g_{N-1}(v), g_{N-1}(v')] \subset T_{N-1}$.

Moreover, because for every N, each of $\iota_N(a)$, $\iota_N(b)$, and $\iota_N(c)$ are vertices of T_N (but the points of the form $\iota(b_t)$, for $t \in [0,1)$, are not), we have:

(2) If v, v' are adjacent vertices in T_N^+, neither of which are vertices in T_N which are mapped into $\{b_t : t \in [0,1)\}$ by the graph-word ρ_N, then $g_{N-1}(v)$ and $g_{N-1}(v')$ are adjacent vertices in T_{N-1}^+.



(a) $N = 1$



(b) $N = 2$

Figure 4.1: Two disjoint intermediate embeddings of G_{ρ_N} in the plane (one in black, one in gray), depicted for the cases $N = 1$ and $N = 2$.

To obtain the next two properties we must take special care when constructing the embeddings Ψ_s. The first is a restatement of the observation described in connection with Figure 4.1. For the second, it should suffice to observe that for every pair of adjacent vertices v, v' in T_N^+, we can make the arc $[v, v']$ the union of two straight line segments in $\mathbb{R}^2$, one from v to p, and one from p to v', where p is some point near the origin.

(3) For each $x \in T_N$, $d(x, g_{N-1}(x)) < \frac{\varepsilon_N}{2} + \frac{1}{2N}$. Moreover, for every pair of adjacent vertices v, v' in T_N^+ as in (2), for each $x \in [v, v']$, $d(x, g_{N-1}(x)) < \frac{\varepsilon_N}{2}$.

(4) For every pair of adjacent vertices v, v' in T_N^+ as in (2), g_N is 1-Lipschitz on $[v, v']$, i.e. $d(g_N(x_1), g_N(x_2)) \leq d(x_1, x_2)$ for every $x_1, x_2 \in [v, v']$.

We may also ensure the following property holds by choosing the sequence $\langle \varepsilon_N \rangle_{n\in\omega}$ to converge to zero fast enough.

(5) If $k \in \omega$ and $\langle (v_N, v'_N) \rangle_{N \geq k}$ and $\langle (w_N, w'_N) \rangle_{N \geq k}$ are two sequences of pairs of adjacent vertices, $v_N, v'_N, w_N, w'_N \in T_N^+$, such that g_N maps $[v_{N+1}, v'_{N+1}]$ onto $[v_N, v'_N]$ and $[w_{N+1}, w'_{N+1}]$ onto $[w_N, w'_N]$, then:

- if $v_N = w_N$ and $[v_N, v'_N] \cap [w_N, w'_N] = \{v_N\}$ for each $N \geq k$, then
$$\limsup_{N\to\infty} [v_N, v'_N] \cap \limsup_{N\to\infty} [w_N, w'_N] = \{\lim_{N\to\infty} v_N\};$$
- if $[v_N, v'_N]$ and $[w_N, w'_N]$ are disjoint for all $N \geq k$, then
$$\limsup_{N\to\infty} [v_N, v'_N] \cap \limsup_{N\to\infty} [w_N, w'_N] = \emptyset.$$

We will now argue that X_y is homeomorphic to $X := \varprojlim \{T_{y\restriction N}, g_N\}$, which is clearly homeomorphic to $\varprojlim \{G_{\rho_N}, f_N\}$ (by definition of the maps g_N).

Observe the following fact:

(∗) If $\langle x_N \rangle_{N\in\omega} \in X$, then there is at most one $N > 0$ for which there are adjacent vertices v, v' in T_N^+ such that $x_N \in [v, v']$ and v is a vertex in T_N which is mapped by ρ_N into $\{b_t : t \in [0, 1)\}$.

This is because for any such N, for each $0 < k < N$, x_k belongs to the arc $[w, w']$, where w, w' are adjacent vertices in T_k^+ and $w = \iota_{T_k}(b)$, hence $\rho_k(w) = b$ and $\rho_k(w') = c$.

From this observation (∗), property (3), and the fact that $\sum_{N\in\omega} \varepsilon_N < \infty$, it follows that if $\langle x_N \rangle_{N\in\omega} \in X$, then $\langle x_N \rangle_{N\in\omega}$ is a Cauchy sequence in $\mathbb{R}^2$. The limit of this sequence belongs to X_y since $X_y = \bigcap_{N\in\omega} \overline{\bigcup_{k\geq N} T_k}$. Hence we can define the map $\Phi : X \to X_y$ by $\Phi(\langle x_N \rangle_{N\in\omega}) = \lim_{N\to\infty} \langle x_N \rangle_{N\in\omega}$. It is straightforward to see that Φ is continuous.

Claim 20.2. Φ is a homeomorphism.

Proof of Claim 20.2. To see that Φ is surjective, let $x \in X_y$, and let $\delta > 0$. Choose k large enough so that $\frac{1}{2k} + \sum_{N \geq k} \frac{\varepsilon_N}{2} < \frac{\delta}{2}$, and such that T_k meets $B_{\frac{\delta}{2}}(x)$ (the $\frac{\delta}{2}$-ball around x). Choose an element $\langle x_N \rangle_{N \in \omega} \in X$ such that $x_k \in B_{\frac{\delta}{2}}(x)$. From property (3) and ($*$), we see that $\Phi(\langle x_N \rangle_{N \in \omega}) = \lim_{N \to \infty} x_N \in B_\delta(x)$. Thus the range of Φ is dense in X_y; since X is compact, it follows that Φ is surjective.

To see that Φ is injective, let $\langle x_N \rangle_{N \in \omega}$ and $\langle x'_N \rangle_{N \in \omega}$ be distinct elements of X. By the observation ($*$), we can choose k large enough so that if $N \geq k$ and v, v' are adjacent vertices in T_N^+ such that $x_N \in [v, v']$ or $x'_N \in [v, v']$, then it is not the case that v is a vertex in T_N which is mapped by ρ_N into $\{b_t : t \in [0,1)\}$.

We consider two cases:

Case 1. For each $N \geq k$, there are adjacent vertices v_N, v'_N in T_N^+ with $x_N, x'_N \in [v_N, v'_N]$.

Since each function g_N maps the arc $[v_{N+1}, v'_{N+1}]$ homeomorphically onto $[v_N, v'_N]$, it follows that $x_k \neq x'_k$. Moreover, by property (4), we have that $d(x_N, x'_N) \geq d(x_k, x'_k)$ for every $N \geq k$. This implies $\Phi(\langle x_N \rangle_{N \in \omega}) \neq \Phi(\langle x'_N \rangle_{N \in \omega})$.

If Case 1 fails, the only other possibility is:

Case 2. There exists $j \geq k$ such that if $N \geq j$ then it is not the case that there are adjacent vertices v, v' in T_N^+ with $x_N, x'_N \in [v, v']$.

Let $\langle v_N, v'_N \rangle_{N \geq j}$ and $\langle w_N, w'_N \rangle_{N \geq j}$ be two sequences of pairs of adjacent vertices, $v_N, v'_N, w_N, w'_N \in T_N^+$, such that $x_N \in [v_N, v'_N]$, $x'_N \in [w_N, w'_N]$, and g_N maps $[v_{N+1}, v'_{N+1}]$ onto $[v_N, v'_N]$ and $[w_{N+1}, w'_{N+1}]$ onto $[w_N, w'_N]$, for each $N \geq j$. Moreover, if $[v_N, v'_N]$ and $[w_N, w'_N]$ have a common endpoint, let it be $v_N = w_N$.

If $v_N = w_N$ for each $N \geq j$, then we cannot have that $x_N = v_N$ or $x'_N = v_N$ (since we are not in Case 1). It follows from Case 1 that $\Phi(\langle x_N \rangle_{N \in \omega}) \neq \lim_{N \to \infty} v_N$ and $\Phi(\langle x'_N \rangle_{N \in \omega}) \neq \lim_{N \to \infty} v_N$. Therefore, by property (5), we see that $\Phi(\langle x_N \rangle_{N \in \omega}) \neq \Phi(\langle x'_N \rangle_{N \in \omega})$.

If instead there is some $j' \geq j$ such that the arcs $[v_N, v'_N]$ and $[w_N, w'_N]$ are disjoint for all $N \geq j'$, then again it follows from property (5) that $\Phi(\langle x_N \rangle_{N \in \omega}) \neq \Phi(\langle x'_N \rangle_{N \in \omega})$. □(Claim 20.2)

The same care used in the choice of embeddings Ψ_s can also be taken in the construction of the example from Chapter 3. Assuming this was done, we have that X

is homeomorphic to the example presented there. In particular, this means X is non-chainable.

Thus we have constructed an uncountable family of pairwise disjoint copies of the non-chainable tree-like continuum X in $\mathbb{R}^2$. We remark that it can be argued directly that what we have constructed is in fact an embedding of $X \times \mathcal{C}$ in $\mathbb{R}^2$; however, we need not bother due to the Theorem of Todorčević quoted above. □

In connection with the above example, the following related question is still open:

Question 4 (Problem 3.8 of [37]). *Is there a tree-like continuum X with positive span which has the property that the plane contains uncountably many pairwise disjoint homeomorphic copies of X?*

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