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Simultaneous Brick Partitionings of Peano Continua

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Property S

Definition

A metric space (M, d) has *property S* if for each $\varepsilon > 0$ there is a finite cover of M by connected sets of diameter less than ε.

- $\mathbb{R}$ in its usual metric does not have property S while $(0, 1)$ in its usual metric does. So this property is a metric property.
- if (M, d) has property S, then it is locally connected and totally bounded.
- every Peano continuum (i.e. locally connected, compact, metric space) has property S.

Partition of a set

Definition

A finite closed covering $\mathcal{V}$ of a metric space (M, d) is a *partition* of M if the following conditions are satisfied for all $U, V \in \mathcal{V}$:

- V and $\mathrm{int}(V)$ are connected and locally connected, and the first one of them is regular closed while the other is regular open,
- if $U \neq V$ then $U \cap V \subset \mathrm{bd}(U) \cap \mathrm{bd}(V)$.

If the mesh of $\mathcal{V}$ is less than $\varepsilon > 0$ then $\mathcal{V}$ is called an ε-partition of M.

Theorem (Bing, 1949)

For a metric space M there is an ε-partitioning for each $\varepsilon > 0$ if and only if M has property S.

Brick partitioning

Definition

We shall say that a partition $\mathcal{V}$ of M is a brick partition of M if the following conditions are satisfied for all $U, V \in \mathcal{V}$:

- $\text{int}(V)$ is uniformly locally connected;
- $\text{int}(U \cup V)$ is uniformly locally connected.

Theorem (Bing, 1949)

Each Peano continuum (i.e. compact, connected and locally connected metric space) has a sequence $\{\mathcal{U}_i\}$ *of brick partitionings such that* $\mathcal{U}_i$ *has mesh less than* $1/i$ *and* $\mathcal{U}_{i+1}$ *refines* $\mathcal{U}_i$.

Core refinements

Let $\mathcal{U}$ and $\mathcal{V}$ be partitions of M such that $\mathcal{V}$ refines $\mathcal{U}$. For each element $U \in \mathcal{U}$ let

$$\mathcal{V}(U) = \{V \in \mathcal{V} : V \subset U\}$$
$$B(U, \mathcal{V}) = \{V \in \mathcal{V}(U) : V \cap \mathrm{bd}(U) \neq \varnothing\}$$
$$I(U, \mathcal{V}) = \mathcal{V}(U) - B(U, \mathcal{V})$$

Definition

We shall say that $\mathcal{V}$ is a *core refinement* of the partition $\mathcal{U}$ of M if $\mathcal{V}$ is a partition of M which refines $\mathcal{U}$ and the following conditions are satisfied for all $U \in \mathcal{U}$:

- $\bigcup I(U, \mathcal{V})$ is connected,
- each member of $B(U, \mathcal{V})$ meets at least one member of $I(U, \mathcal{V})$.

Simultaneous partitioning for two Peano continua

Definition

Let $\mathcal{U}$ be a partition of the connected, locally connected, metric space M and let N be a closed connected subset of M such that $\mathcal{U}|_N = \{U \cap N \mid U \in \mathcal{U}\}$ is a partition of N. Then $\mathcal{U}$ is called a simultaneous partition of M and N.

Let $\mathcal{V}$ be a core refinement of $\mathcal{U}$ such that $\mathcal{V}|_N = \{V \cap N \mid V \in \mathcal{V}\}$ is a core refinement of $\mathcal{U}|_N$. Then we say that $\mathcal{V}$ is a simultaneous core refinement of $\mathcal{U}$ with respect to M and N.

Theorem (Thomas, 1953)

Let M be a Peano continuum and N a Peano subcontinuum of M. Then there exists a sequence of simultaneous core partitionings $\mathcal{U}_i$ of M and N such that $\mathcal{U}_i$ has mesh less than $1/i$ and $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$.

Simultaneous brick partitioning of two Peano continua

Theorem (E.D. Tymchatyn, I.S.)

Let $N \subset M$ be Peano continua. Then there is a sequence $\{\mathcal{U}_i\}$ of simultaneous brick partitions of M and N such that $\mathcal{U}_{i+1}$ is a refinement of $\mathcal{U}_i$ and mesh $\mathcal{U}_i$ is less than $1/i$ for each i.

Theorem (E.D. Tymchatyn, I.S.)

Let M and N be Peano continua with $N \subset M$. Suppose that $\mathcal{U}$ is a simultaneous brick partition of M and N. Then for each $\varepsilon > 0$ there exists a simultaneous core brick partition $\mathcal{V}$ with respect to M and N such that $\mathcal{V}$ refines $\mathcal{U}$ and $\mathrm{mesh}\mathcal{V} < \varepsilon$.

Simultaneous brick partitioning of two Peano continua

Comments on proof. Let $\varepsilon > 0$ and let $\mathcal{U}'$ be a simultaneous brick ε-partition of M and N which refines $\mathcal{U}$.

Let T be a connected graph in M such that

1) $T \cap V \cap N$ is a tree for each $V \in \mathcal{U}'$ with $V \cap N \neq \varnothing$ and $\mathrm{bd}_N(V \cap N) \cap T$ contains exactly one endpoint of $T \cap N \cap V$;
2) $T \cap V$ is a tree for each $V \in \mathcal{U}'$ and $T \cap \mathrm{bd}(V)$ contains exactly one endpoint of $T \cap V$;
3) if $V, V' \in \mathcal{U}'$ are adjacent and $V \cap N \neq \varnothing \neq V' \cap N$ then there is an arc A in T such that $A \subset \mathrm{int}(V \cup V')$ which meets both $\mathrm{int}_N(V \cap N)$ and $\mathrm{int}_N(V' \cap N)$ and $A \cap V \cap V'$ is a singleton;
4) if $V, V' \in \mathcal{U}'$ are adjacent then there is an arc $A \subset \mathrm{int}(V \cup V')$ which meets both $\mathrm{int}(V)$ and $\mathrm{int}(V')$ and $A \cap V \cap V'$ is a singleton;

Simultaneous brick partitioning of two Peano continua

For each $U \in \mathcal{U}$ let $T_U \subset T \cap \operatorname{int}(U)$ be a compact connected graph such that $(T \cap U) \setminus T_U$ consists of half open arcs $A_{U,1}, \ldots, A_{U,n_U}$ with pairwise disjoint closures and such that each $A_{U,i} \cap \operatorname{bd}(U)$ is the singleton end-point of $A_{U,i}$. Note that $V \cap T_U$ is connected for each $V \in \mathcal{U}'(U)$. Let

$$0 < \delta < \frac{1}{2} \min \Big(\big\{ d(T_U, \operatorname{bd}(U)) \mid U \in \mathcal{U}' \big\} \cup$$

$$\cup \big\{ d(A_{U,i}, A_{U,j}) \mid U \in \mathcal{U},\ i \neq j \in \{1, \ldots, n_U\} \big\} \Big).$$

Simultaneous brick partitioning of two Peano continua

Let $\mathcal{U}''$ be a simultaneous brick δ-partition of M and N which refines $\mathcal{U}'$. For each $U \in \mathcal{U}$ let U_c be the largest connected set in $\mathrm{int}(U)$ which contains T_U and consists of a union of elements of $\mathcal{U}''$.

Let $U \in \mathcal{U}$.
If $V \in I(U, \mathcal{U}')$ then $V \subset U_c$ since V is the connected union of members of $I(U, \mathcal{U}'')$ and $V \cap T_U \neq \varnothing$. Set $V \in \mathcal{V}$.

If $V \in B(U, \mathcal{U}')$ let V_c be the component of $T_U \cap V$ in $U_c \cap V$.
Let $V_c \in I(U, \mathcal{V})$.
Then each component of $V \setminus V_c$ meets $\mathrm{bd}(U)$.
Let $K_{V,1}, \ldots, K_{V,m_V}$ be the components of $(V \cap N) \setminus \mathrm{int}_V(V_c)$.
For each $i \in \{1, \ldots, m_V\}$ let

$$S(K_{V,i}) = \bigcup \mathrm{st}(K_{V,i}, \mathcal{U}''(V \setminus \mathrm{int}_V(V_c)).$$

Let

$$M_V = V \setminus \mathrm{int}_V\Big(V_c \cup \bigcup\{S(K_{V,i})\colon i = 1, \ldots, m_V\}\Big).$$

Simultaneous brick partitioning of two Peano continua

Let L be a component of M_V.

If $L \cap V_c \neq \varnothing$ set $L \in \mathcal{V}$.

If $L \cap V_c = \varnothing$ then $L \cap S(K_{V,i}) \neq \varnothing$ for some smallest i.

For each $i = 1, \ldots, m_V$ set

$$\bigcup \Big\{ L \text{ is a component of } M_V \mid L \cap V_c = \varnothing \text{ and }$$
$$i \text{ is smallest such that } S(K_{V,i}) \cap L \neq \varnothing \Big\} \cup S(K_{V,i}).$$

Then $\mathcal{V}$ is the required simultaneous ε-brick partition with respect to M and N which core refines $\mathcal{U}$.

THANK YOU !