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Simultaneous Extensions of Convex Metrics

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Preliminaries

Definition

A metric r on a metric space X is said to be *convex* if for each $x, y \in X$ there is an arc $[xy]$ with endpoints x and y such that $[xy]$ is isometric to the closed interval $[0, r(x, y)]$ in the real line $\mathbb{R}$. We call such an arc an r-segment.

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- Tominaga and Tanaka showed that every locally connected, locally compact, separable, metric space admits a convex metric (1955).
- Nikiel, Tuncali, Tymchatyn and S. proved that every connected, locally arc-connected, metric space with property S admits a convex metric (2013).

Extending convex metrics

Theorem (Bing, 1949)

If M_1 and M_2 are two intersecting Peano continua with convex metrics d_1 and d_2 respectively, there is a convex metric d_3 on $M_1 \cup M_2$ that preserves d_1 on M_1.

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Theorem (Dooley, 1973)

Let M_1 be a space with a complete convex metric d_1 and let M_2 be a locally connected, locally compact, separable, metric space. In order for there to be a complete convex metric for $M_1 \cup M_2$ that extends d_1, it is necessary and sufficient that $M_1 \cap M_2$ be a nonempty subspace of both M_1 and M_2 which is closed in M_2 and whose M_2 boundary of $M_1 \cap M_2$ be closed in M_1.

Main result

Theorem (E.D. Tymchatyn and I.S.)

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$$\mathcal{CM} = \bigcup \{\mathcal{CM}(A) : A \text{ is a Peano subcontinuum of } X\}.$$

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- We write $\mathrm{dom}\rho = \mathrm{A}$ if $\rho \in \mathcal{CM}(A)$.

Space of partial convex metrics

- We will identify each metric $\rho \in \mathcal{CM}$ with its graph

$$\Gamma_\rho = \{(x, y, \rho(x, y))\colon x, y \in \mathrm{dom}\rho\}$$

which is a compact subset of $X \times X \times \mathbb{R}$.

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- Let r be a fixed compatible convex metric on X with $\|r\|_\infty = 1$.
- Let the metric r' on $X \times X \times \mathbb{R}$ be defined by

$$r'[(x, y, z), (x', y', z')] = r(x, x') + r(y, y') + |z - z'|$$

for every $x, y, x', y' \in X$ and $z, z' \in \mathbb{R}$.

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  for every $x, y, x', y' \in X$ and $z, z' \in \mathbb{R}$.
- The topology on $\mathcal{CM}$ is the Hausdorff metric topology generated by the metric r' under the assumption that each metric $\rho \in \mathcal{CM}$ is identified with its graph.

Extension operator

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- For $\rho \in \mathcal{CM}$ let $\varphi(\rho)$ be the smallest concave modulus function for ρ that is $\varphi(\rho)\colon [0,1] \to [0,+\infty)$ is the least concave function such that

$$\varphi(\rho)(t) \geq \max\{\rho(x,y) : x, y \in \mathrm{dom}\rho,\ r(x,y) \leq t\}.$$

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- $\varphi(\rho)$ is nondecreasing.
- $\varphi(\rho)'_-(t) = \varphi(\rho)'_+(t) = \varphi(\rho)'(t)$ for all but countably many $t \in (0,1)$ and $\varphi(\rho)'_-(t)$ is nonincreasing.

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- Define a continuous function $\theta\colon \mathcal{CM} \to C([0,1])$ by

$$\theta(\rho)(t) = \begin{cases} \varphi(\rho)(t) + t(1 - \varphi(\rho)'_-(1)) \text{ if } \varphi(\rho)'_-(1) < 1; \\ \varphi(\rho)(t) \text{ if } \varphi(\rho)'_-(1) \geq 1 \end{cases}$$

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- There is a non-increasing function $\nu\colon [0,1] \to [0,\infty)$ such that $\theta(\rho)(t_0) = \int_0^{t_0} \nu(t)dt$ for $\rho \in \mathcal{CM}$ and $t_0 \in [0,1]$.

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- Then $\theta(\rho)$ is a concave modulus function for the metric ρ with $\theta(\rho)(0) = 0$ and $\theta(\rho)'_-(t) \geq 1$ for all $t \in (0,1]$.
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- In fact $\nu(t) = \theta(\rho)'_-(t)$ almost everywhere on $[0,1]$.
- Therefore

$$\int_0^{t_0} \theta(\rho)'_-(t)dt = \theta(\rho)(t_0) \text{ for } \rho \in \mathcal{CM} \text{ and } t_0 \in [0,1].$$

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- For $\rho \in \mathcal{CM}$ let $\mathcal{A}_\rho$ denote the collection of all r-rectifiable paths which meet $\mathrm{dom}\rho$ in at most their endpoints.
- For $C \in \mathcal{A}_\rho$ define the ρ-length of C by

$$L_\rho(C) = \int_C \theta(\rho)'_-(r(p(s), \mathrm{dom}\rho))ds$$

where C is parametrized by its arc length with respect to r and $p(s)$ is a point on C whose distance along C from a fixed endpoint of C is s.

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  where C is parametrized by its arc length with respect to r and $p(s)$ is a point on C whose distance along C from a fixed endpoint of C is s.
- Note that

$$L_\rho(C) \geq L_r(C)$$

  since $\theta(\rho)'_- \geq 1$ and $\theta(r)$ is the identity on $[0, 1]$.

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$$\sigma_\rho(x,y) = \inf\{L_\rho(C) : C \in \mathcal{A}_\rho(x,y)\}$$

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  and let $\sigma_\rho(x,y) = \infty$ if $\mathcal{A}_\rho(x,y) = \varnothing$.
- Then $\sigma_\rho(x,y) \geq r(x,y)$ because $L_\rho(C) \geq L_r(C)$.

Extension operator

- Let $x, y \in \mathrm{dom}\rho$ with $\sigma_\rho(x,y) < \infty$ and fix $\varepsilon > 0$. There exists $C \in \mathcal{A}_\rho(x,y)$ such that

$$\sigma_\rho(x,y) + \varepsilon > L_\rho(C) = \int_C \theta(\rho)'_-(r(p(s), \mathrm{dom}\rho))ds \geq \int_C \theta(\rho)'_-(s)ds = \int_0^{L_r(C)} \theta(\rho)'_-(s)ds = \theta(\rho)(L_r(C)) \geq \theta(\rho)(r(x,y)) \geq \rho(x,y).$$

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- The first $\geq$ above is because $\theta(\rho)'_-$ is non-increasing and $r(p(s), \mathrm{dom}\rho) \leq s$ since the endpoints of C lie in $\mathrm{dom}\rho$.

  The second $\geq$ above is because $\theta(\rho)$ is non-decreasing.

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- Since ε is arbitrary, we obtain $\sigma_\rho \geq \rho$ on $\mathrm{dom}\rho$.

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- For $x, y \in X$ define

$$u(\rho)(x,y) = \begin{cases} \rho(x,y) \text{ if } x,y \in \mathrm{dom}\rho; \\ \alpha_\rho(x,y) \text{ if } |\{x,y\} \cap \mathrm{dom}\rho| = 1; \\ \min\{\sigma_\rho(x,y), \beta_\rho(x,y)\} \text{ if } x,y \in X \setminus \mathrm{dom}\rho \end{cases}$$

  where

$$\alpha_\rho(b,c) = \alpha_\rho(c,b) = \inf\{\sigma_\rho(c,a) + \rho(a,b) : a \in \mathrm{dom}\rho\}$$

  for $b \in \mathrm{dom}\rho$, $c \in X \setminus \mathrm{dom}\rho$ and

$$\beta_\rho(x,y) = \inf\{\sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(b,y) : a,b \in \mathrm{dom}\rho\}.$$

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- This definition of $u(\rho)(x, y)$ is equivalent to defining $u(\rho)(x, y)$ to be the greatest lower bound of lengths of all arcs C from x to y where length in $\mathrm{dom}\rho$ is measured by ρ and length in $X \setminus \mathrm{dom}\rho$ is measured by L_ρ.

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and $u(\rho)$ is continuous with respect to r.

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- $u(\rho)$ is a convex metric on X such that

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  and $u(\rho)$ is continuous with respect to r.
- For every sequence $\{\rho_i\}$ in $\mathcal{CM}$ that converges to $\rho \in \mathcal{CM}$, the sequence of extensions, $\{u(\rho_i)\}$ converges to $u(\rho)$ in $\mathcal{CM}(X)$.

THANK YOU