On Existence of Convex Metrics on Non-Compact Spaces

I.Stasyuk
(with J. Nikiel, M.Tuncali and E.D.Tymchatyn)

Nipissing University

ihors@nipissingu.ca

March 25, 2013

Introduction

Definition

Let (X, ρ) be a metric space. We say that ρ is *convex* if for each $x \neq y \in X$ there is an arc $A \subset X$ with end-points x and y such that $(A, \rho|_{A \times A})$ is isometric to the interval $[0, \rho(x, y)]$ in the real line.

Theorem (Bing, 1949)

Every Peano continuum (i.e. compact, connected and locally connected metric space) admits a compatible convex metric.

- $\{\mathcal{U}_n\}_{n=1}^{\infty}$ - a decreasing sequence of finite partitions of X.
- $w_n(U)$ - the weight assigned to $U \in \mathcal{U}_n$. A subcollection of $\mathcal{U}_n$ has weight equal to the sum of the weights of its elements.
- An approximation to the distance $\rho(x, y)$ is the smallest of the weights $w_n(\mathcal{C}_n)$ of chains $\mathcal{C}_n$ in $\mathcal{U}_n$ from x to y.
- Then $\limsup \bigcup \mathcal{C}_n$ contains a line segment (with respect to ρ) from x to y.

Introduction

Bing asked for an extension of his result to non-compact metric spaces. In the non-compact case, $\limsup \bigcup \mathcal{C}_n$ need not in general contain a connected subset from x to y.

Consider for instance $X = ([0,1] \times [0,1]) \setminus ((0,1) \times \{0\})$ in its usual metric inherited from the plane and choose $x = (0,0)$ and $y = (1,0)$.

Definition

A metric space (X, ρ) has *property S* if for each $\varepsilon > 0$ there is a finite cover of X by connected sets of diameter less than ε.

- $\mathbb{R}$ in its usual metric does not have property S while $(0,1)$ in its usual metric does. So this property is a metric property.
- if (X, d) has property S, then it is locally connected and totally bounded.
- every locally connected metric continuum has property S.
- there exist locally connected, connected, separable metric spaces which do not have property S in any metric.

Constructing a convex metric

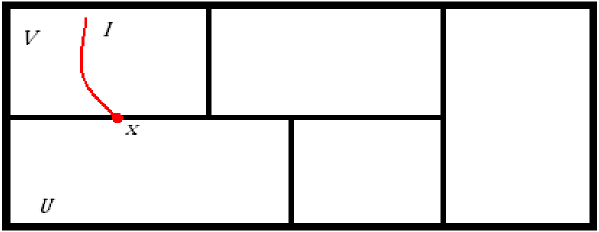
Theorem (J.Nikiel, M.Tuncali, E.D. Tymchatyn, I.S.)

If X is a connected and locally arc-connected metric space with property S, then X admits a convex metric.

Definition

A closed covering $\mathcal{V}$ of X is a *partition* of X if the following conditions are satisfied for all $U, V \in \mathcal{V}$:

- V and int(V) are connected and locally connected, V - regular closed, int(V) - regular open,
- if $U \neq V$ then $U \cap V \subset \mathrm{bd}(U) \cap \mathrm{bd}(V)$,
- if $U \cap V \neq \varnothing$ then int($U \cup V$) is connected, and
- bd(V) is accessible from int(V), i.e., for each $x \in \mathrm{bd}(V)$ there exists an arc I such that $x \in I \subset \{x\} \cup \mathrm{int}(V)$.

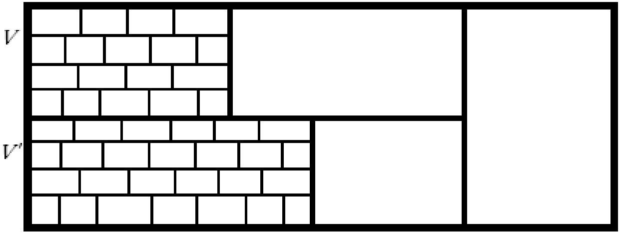


Constructing a convex metric

Definition

A collection $\mathcal{U}$ is a *core refinement* of a partition $\mathcal{V}$ of X if $\mathcal{U}$ is also a partition of X and the following conditions are satisfied for all $V, V' \in \mathcal{V}$:

- the union of all interior elements from $\mathcal{U}$ contained in V is connected,
- each boundary element from $\mathcal{U}$ meets an interior element from $\mathcal{U}$.
- if $V \cap V' \neq \varnothing$ then the union of interior elements from $\mathcal{U}$ that are contained in $V \cup V'$, is connected.

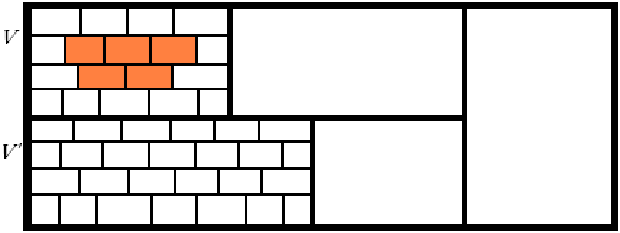


Constructing a convex metric

Definition

A collection $\mathcal{U}$ is a *core refinement* of a partition $\mathcal{V}$ of X if $\mathcal{U}$ is also a partition of X and the following conditions are satisfied for all $V, V' \in \mathcal{V}$:

- the union of all interior elements from $\mathcal{U}$ contained in V is connected,
- each boundary element from $\mathcal{U}$ meets an interior element from $\mathcal{U}$.
- if $V \cap V' \neq \varnothing$ then the union of interior elements from $\mathcal{U}$ that are contained in $V \cup V'$, is connected.

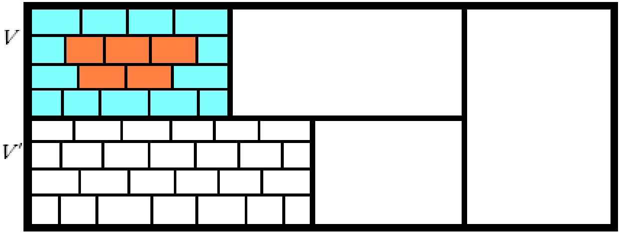


Constructing a convex metric

Definition

A collection $\mathcal{U}$ is a *core refinement* of a partition $\mathcal{V}$ of X if $\mathcal{U}$ is also a partition of X and the following conditions are satisfied for all $V, V' \in \mathcal{V}$:

- the union of all interior elements from $\mathcal{U}$ contained in V is connected,
- each boundary element from $\mathcal{U}$ meets an interior element from $\mathcal{U}$.
- if $V \cap V' \neq \varnothing$ then the union of interior elements from $\mathcal{U}$ that are contained in $V \cup V'$, is connected.

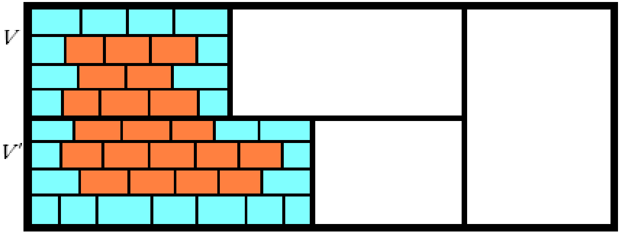


Constructing a convex metric

Definition

A collection $\mathcal{U}$ is a *core refinement* of a partition $\mathcal{V}$ of X if $\mathcal{U}$ is also a partition of X and the following conditions are satisfied for all $V, V' \in \mathcal{V}$:

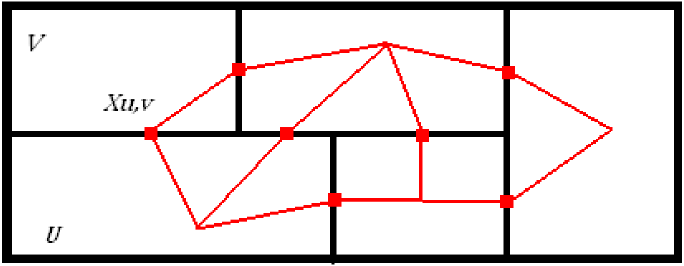
- the union of all interior elements from $\mathcal{U}$ contained in V is connected,
- each boundary element from $\mathcal{U}$ meets an interior element from $\mathcal{U}$.
- if $V \cap V' \neq \varnothing$ then the union of interior elements from $\mathcal{U}$ that are contained in $V \cup V'$, is connected.



Constructing a convex metric

For each space (X, d) which is connected, locally arc-connected and has property S, there exists a finite partition $\mathcal{U}_1$ of X of mesh less than 1 such that $\mathrm{int}(U)$ has property S for every $U \in \mathcal{U}_1$.

- For all $U, V \in \mathcal{U}_1$ which are adjacent fix $x_{U,V} \in \mathrm{int}(U \cup V) \cap \mathrm{bd}(U)$.
- For each $V \in \mathcal{U}_1$ let G_V be a finite and simply connected graph in V such that $G_V \cap \mathrm{bd}(V) = \{x_{U,V} : U \in \mathcal{U}_1 \text{ is adjacent to } V\}$.
- Let $G_1 = \bigcup\{G_V : V \in \mathcal{U}_1\}$.



Constructing a convex metric

Inductively, we construct a family $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of partitions of X, and a sequence $\{G_i\}_{i=1}^{\infty}$ of finite graphs in X such that

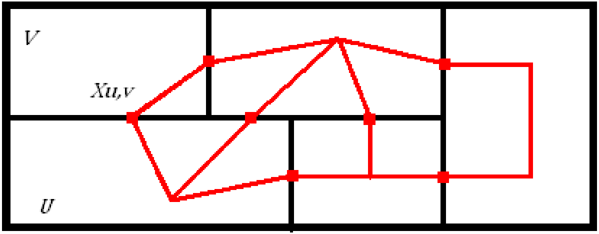
(a) $\operatorname{mesh} \mathcal{U}_i < 2^{-i}$,

(b) $\operatorname{int}(U)$ has property S for each $U \in \mathcal{U}_i$,

(c) $\mathcal{U}_{i+1}$ is a core refinement of $\mathcal{U}_i$,

(d) $G_i \subset G_{i+1}$,

(g) if $W \in \mathcal{U}_{i+1}$ then $G_i \cap \operatorname{int}(W)$ is connected and $G_i \cap \operatorname{bd}(W) \subset \operatorname{cl}(G_i \cap \operatorname{int}(W))$.



Constructing a convex metric

Inductively, we construct a family $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of partitions of X, and a sequence $\{G_i\}_{i=1}^{\infty}$ of finite graphs in X such that

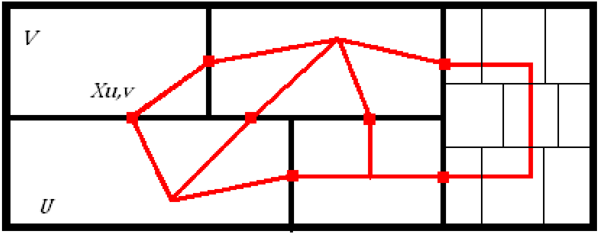
(a) $\mathrm{mesh}\,\mathcal{U}_i < 2^{-i}$,

(b) $\mathrm{int}(U)$ has property S for each $U \in \mathcal{U}_i$,

(c) $\mathcal{U}_{i+1}$ is a core refinement of $\mathcal{U}_i$,

(d) $G_i \subset G_{i+1}$,

(g) if $W \in \mathcal{U}_{i+1}$ then $G_i \cap \mathrm{int}(W)$ is connected and $G_i \cap \mathrm{bd}(W) \subset \mathrm{cl}(G_i \cap \mathrm{int}(W))$.



Constructing a convex metric

Inductively, we construct a family $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of partitions of X, and a sequence $\{G_i\}_{i=1}^{\infty}$ of finite graphs in X such that

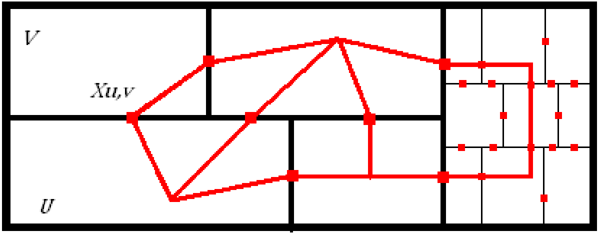
(a) $\operatorname{mesh} \mathcal{U}_i < 2^{-i}$,

(b) $\operatorname{int}(U)$ has property S for each $U \in \mathcal{U}_i$,

(c) $\mathcal{U}_{i+1}$ is a core refinement of $\mathcal{U}_i$,

(d) $G_i \subset G_{i+1}$,

(g) if $W \in \mathcal{U}_{i+1}$ then $G_i \cap \operatorname{int}(W)$ is connected and $G_i \cap \operatorname{bd}(W) \subset \operatorname{cl}(G_i \cap \operatorname{int}(W))$.



Constructing a convex metric

Inductively, we construct a family $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of partitions of X, and a sequence $\{G_i\}_{i=1}^{\infty}$ of finite graphs in X such that

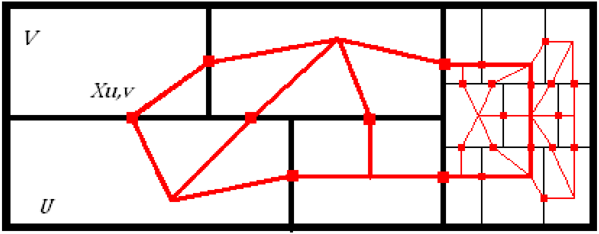
(a) $\operatorname{mesh} \mathcal{U}_i < 2^{-i}$,

(b) $\operatorname{int}(U)$ has property S for each $U \in \mathcal{U}_i$,

(c) $\mathcal{U}_{i+1}$ is a core refinement of $\mathcal{U}_i$,

(d) $G_i \subset G_{i+1}$,

(g) if $W \in \mathcal{U}_{i+1}$ then $G_i \cap \operatorname{int}(W)$ is connected and $G_i \cap \operatorname{bd}(W) \subset \operatorname{cl}(G_i \cap \operatorname{int}(W))$.



Constructing a convex metric

Definition

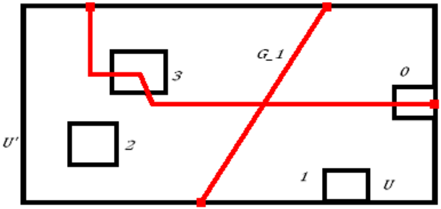
For each $U \in \mathcal{U}_1$ define the weight $w_1(U) = 1$. For each connected subset K of X such that $K = \bigcup \mathcal{V}$ for some subcollection $\mathcal{V}$ of $\mathcal{U}_1$, we say that K is a set which has 1^{st}-weight and we set $w_1(K) = \sum_{V \in \mathcal{V}} w_1(V)$.

Constructing a convex metric

Definition

For $U \in \mathcal{U}_2$ the *position* of U is an integer $P(U) \in \{0,1,2,3\}$ such that if $U \subset U' \in \mathcal{U}_1$ then

$$P(U) = \begin{cases} 0 \text{ if } U \text{ is boundary in } U' \text{ and } U \cap G_1 \neq \varnothing \\ 1 \text{ if } U \text{ is boundary in } U' \text{ and } U \cap G_1 = \varnothing \\ 2 \text{ if } U \text{ is interior in } U' \text{ and } U \cap G_1 = \varnothing \\ 3 \text{ if } U \text{ is interior in } U' \text{ and } U \cap G_1 \neq \varnothing. \end{cases}$$



Constructing a convex metric

Definition

Suppose that for an integer $n \geq 2$ the position $P(U')$ of U' is defined to be an ordered $(n-1)$-tuple of integers, for each $U' \in \mathcal{U}_n$. Let $U \in \mathcal{U}_{n+1}$ and $U' \in \mathcal{U}_n$ be such that $U \subset U'$. Define the *position* $P(U)$ of V to be the ordered n-tuple $(k_1, k_2, \ldots, k_n)$, where $(k_1, k_2, \ldots, k_{n-1}) = P(U')$ and

$$k_n = \begin{cases} 0 \text{ if } U \text{ is boundary in } U' \text{ and } U \cap G_n \neq \varnothing \\ 1 \text{ if } U \text{ is boundary in } U' \text{ and } U \cap G_n = \varnothing \\ 2 \text{ it } U \text{ is interior in } U' \text{ and } U \cap G_n = \varnothing \\ 3 \text{ if } U \text{ is interior in } U' \text{ and } U \cap G_n \neq \varnothing. \end{cases}$$

Constructing a convex metric

Definition

Let $\delta_2 = \frac{1}{4}$. For $U \in \mathcal{U}_2$ let

$$w_2(U) = \begin{cases} 2^{-1} \text{ if } P(U) = 0 \\ 2^{-1} + \delta_2/|\mathcal{U}_2|^2 \text{ if } P(U) = 1 \\ \delta_2/|\mathcal{U}_2|^{2j} \text{ if } P(U) = j \in \{2, 3\}. \end{cases}$$

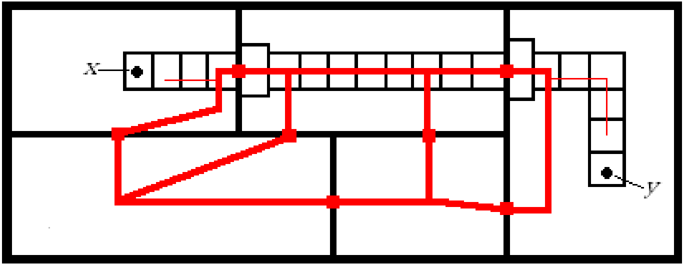
Suppose that the numbers $\delta_i > 0$ and $w_i(U)$ are defined for each integer i with $2 \leq i \leq n$ and each $U \in \mathcal{U}_i$. Choose a sufficiently small δ_{n+1}. For each $U \in \mathcal{U}_{n+1}$ let $U' \in \mathcal{U}_n$ be such that $U \subset U'$ and let

$$w_{n+1}(U) = \begin{cases} w_n(U')/2 \text{ if } P(U) = (P(U'), 0) \\ w_n(U')/2 + \delta_{n+1}/|\mathcal{U}_{n+1}|^2 \text{ if } P(U) = (P(U'), 1) \\ \delta_{n+1}/|\mathcal{U}_{n+1}|^{2j} \text{ if } P(U) = (P(U'), k) \text{ for } j \in \{2, 3\}. \end{cases}$$

Constructing a convex metric

Definition

- If K is a non-empty connected subset of X such that $K = \bigcup \mathcal{V}$ for some subcollection $\mathcal{V}$ of $\mathcal{U}_i$, then K has i^{th}-*weight* and $w_i(K) = \sum_{v \in \mathcal{V}} w_i(V)$.
- Let $E_i(x, y) = \min\{w_i(K) \mid K \subset X \text{ has } i^{\text{th}} \text{ weight and } x, y \in K\}$. Then $E_i(x, y)$ is the i^{th} approximation of the distance from x to y.



References

R.H.Bing, *Partitioning a set*, Bull. Amer. Math. Soc. **55** (1949), 1101–1110.

R.H.Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. **58** (1952), 536–556.

K. Borsuk, *On intrinsic isometries*, Bull. Acad. Pol. Sci. **29** (1981), 83–90.

A.Chigogidze, K.Kawamura and E.D.Tymchatyn, *Nöbeling spaces and pseudo-interiors of Menger compacta*, Topology Appl. **68** (1996), 33-65.

V.V.Filippov, *On locally connected spaces*, Dokl. Akad. Nauk SSSR **215** (1974), 666–668.

E.E.Moise, *Grille decomposition and convexification theorems for compact metric locally connected continua*, Bull. Amer. Math. Soc. **55** (1949), 1111–1121.

G.H.M.Thomas, *Simultaneous partitionings of two sets*, Trans. Amer. Math. Soc. **75** (1953), 69–79.

THANK YOU