# EXTENSION OF UNIFORMLY DISCONNECTED METRICS AND ULTRAMETRICS

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**Definition.** *A metric $r$ on a set $Y$ is called an ultrametric if it satisfies the strong triangle inequality* $r(x,y) \le \max\{r(x,z), r(y,z)\}$ *for all* $x,y,z \in Y$.

$\exp Y = \{A \subset Y :\ A \text{ is closed and bounded in} (Y,r)\}$ with the Hausdorff metric $H_r$.

$(X,d)$ — a bounded complete ultrametric space.

$\mathcal{UM}(A)$ (respectively $\mathcal{UM}_{\mathrm{u}}(A)$) — the family of all bounded continuous (respectively bounded uniformly continuous) ultrametrics on $A$, $A \in \exp X$, $|A| \ge 2$.

$$\mathcal{UM} = \bigcup\{\mathcal{UM}(A) : A \in \exp X,\ |A| \ge 2\}$$

$$\mathcal{UM}_{\mathrm{u}} = \bigcup\{\mathcal{UM}_{\mathrm{u}}(A) : A \in \exp X,\ |A| \ge 2\}.$$

$\mathrm{dom}\rho = A$ if $\rho \in \mathcal{UM}(A)$.

Each ultrametric $\rho \in \mathcal{UM}$ is identified with its graph

$$\rho \to \Gamma_\rho = \{(x, y, \rho(x, y)) : x, y \in \mathrm{dom}\rho\} \in \exp(X \times X \times \mathbb{R})$$

The set $\exp(X \times X \times \mathbb{R})$ is equipped with the Hausdorff metric $H_{d'}$ generated by the metric $d'$ on $X \times X \times \mathbb{R}$ where

$$d'[(x, y, z), (x_1, y_1, z_1)] = \max\{d(x, x_1), d(y, y_1), |z - z_1|\}.$$

$\mathcal{UM} \subset \exp(X \times X \times \mathbb{R}, H_{d'}), \quad \mathcal{UM}_{\mathrm{u}} \subset \mathcal{UM}$

$\|\rho\| = \max\{\rho(x, y) : x, y \in \mathrm{dom}\rho\}$, $\rho \in \mathcal{UM}$.

**Definition.** *Let $\alpha, \beta \geq 0$. A metric space $(Y, r)$ is called $(\alpha, \beta)$-homogeneous if for every $a, b > 0$ and $B \subset Y$ such that $a \leq r(x, y) \leq b$ whenever $x, y \in Y$ and $x \neq y$, we have $|B| \leq \alpha(a/b)^\beta$. The space $(Y, r)$ is called $\beta$-homogeneous if it is $(\alpha, \beta)$-homogeneous for some $\alpha \geq 0$. We define the Assouad dimension $\dim_A(Y, r)$ of the space $(Y, r)$ as $\dim_A(Y, r) = \inf\{\beta \geq 0 : (Y, r)$ is $\beta$ homogeneous $\}$.*

**Theorem** (E.D. Tymchatyn and M. Zarichnyi). *Let $(X, d)$ be a compact ultrametric space. There exists an operator $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ that satisfies the following conditions for every $\rho, \rho_1 \in \mathcal{UM}$ and $c > 0$*

1) $u(\rho)$ *is an extension of* $\rho$ *over* $X$*;*

2) $u$ *is positive-homogeneous i.e.* $u(c\rho) = cu(\rho)$*;*

3) $u(\max\{\rho, \rho_1\}) = \max\{u(\rho), u(\rho_1)\}$ *if* $\mathrm{dom}\rho = \mathrm{dom}\rho_1$*;*

4) $\|u(\rho)\| = \|\rho\|$*;*

5) $u$ *is a continuous operator;*

6) $\dim_A(X, u(\rho)) = \dim_A(\mathrm{dom}\rho, \rho)$.

**Theorem 1.** *Let $(X,d)$ be a bounded complete ultrametric space. There exists an operator $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ that satisfies the following conditions for every $\rho, \rho_1 \in \mathcal{UM}$ and $c > 0$.*

1) *$u(\rho)$ is an extension of $\rho$ over $X$;*

2) *$u$ is positive-homogeneous i.e. $u(c\rho) = cu(\rho)$;*

3) *$u(\max\{\rho, \rho_1\}) = \max\{u(\rho), u(\rho_1)\}$ if $\mathrm{dom}\rho = \mathrm{dom}\rho_1$;*

4) *$\|u(\rho)\| = \|\rho\|$;*

5) *If $\{\rho_n\}$ is a sequence in $\mathcal{UM}$ such that $\{\Gamma_{\rho_n}\}$ converges to $\Gamma_\rho$ for some $\rho \in \mathcal{UM}$ then $\{u(\rho_n)\}$ converges to $u(\rho)$ pointwise on $X \times X$;*

*Comments on proof.*

$$u(\rho)(x,y) =$$

$$\max\left\{\rho(f(x,\mathrm{dom}\rho), f(y,\mathrm{dom}\rho)), \|\rho\| \max_{i\in\mathbb{N}_+}\left\{2^{-i}w_i(\rho)(x,y)\right\}\right\}$$

for $\rho \in \mathcal{UM}$ and $x, y \in X$. $f\colon X \times \exp X \to X$ is uniformly continuous. $f(x,A) \in A$ for all $(x,A) \in X \times \exp X$ and $f(x,A) = x$ whenever $x \in A$. $\{w_i(\rho)\}_{i\in\mathbb{N}_+}$ – the family of ultrapseudometrics which separates the points of the space $X$ and depends on $\rho$.

□

**Theorem 2.** *There exists an operator* $v\colon \mathcal{UM}_{\rm u} \to \mathcal{UM}_{\rm u}(X)$ *which has properties* 1), 2), 3) *and* 4) *from Theorem 1. Moreover, if* $\{\rho_n\}$ *is a sequence in* $\mathcal{UM}_{\rm u}$ *such that* $\{\Gamma_{\rho_n}\}$ *converges to* $\Gamma_\rho$ *for some* $\rho \in \mathcal{UM}_{\rm u}$ *then* $\{v(\rho_n)\}$ *converges to* $v(\rho)$ *uniformly on* $X \times X$*;*

**Theorem 3.** *Let* $(X, d)$ *be a separable complete ultrametric space. Then there exists a map* $u\colon \mathcal{UM}_{\rm u} \to \mathcal{UM}_{\rm u}(X)$ *that satisfies the following conditions for every* $\rho, \rho_1 \in \mathcal{UM}$ *and* $c > 0$.

1) $v(\rho)$ *is an extension of* $\rho$ *over* $X$*;*

2) $v(c\rho) = cv(\rho)$*;*

3) $v(\max\{\rho, \rho_1\}) = \max\{v(\rho), v(\rho_1)\}$ *if* $\mathrm{dom}\rho = \mathrm{dom}\rho_1$*;*

4) $\|v(\rho)\| = \|\rho\|$*;*

5) $v$ *is a continuous map with respect to the Hausdorff metric topology on* $\mathcal{UM}_{\rm u}$ *and on* $\mathcal{UM}_{\rm u}(X)$.

6) $\dim_A(X, v(\rho)) = \dim_A(\mathrm{dom}\rho, \rho)$.

*Comments on proof.*

$$v(\rho)(x,y) =$$

$$\max\{\rho(f(x,\mathrm{dom}\rho), f(y,\mathrm{dom}\rho)), \|\rho\| r(g(x,\mathrm{dom}\rho), g(y,\mathrm{dom}\rho))\}$$

$g\colon X \times \exp X \to (C,r)$.

$(C,r)$ – the Cantor set with a metric $r$ such that $\dim_A(C,r) = 0$.

$g(x,A) = g(y,A)$ if $x, y \in A$

$g(x,A) \neq g(y,A)$ if $x \neq y$ and $x \notin A \vee y \notin A$ . □

**Definition.** *A metric space* $(Y,\rho)$ *is called c-uniformly disconnected, if there exists a constant* $c > 0$ *such that*

$$c\rho(y_0, y_n) \leq \max\{\rho(y_{j-1}, y_j) : j \in \{1,\dots,n\}\}$$

*for every finite chain* $y_0, y_1, \dots, y_n$ *of elements of the set* $Y$*.*

$\mathcal{UDM}(A)$ – the set of continuous uniformly disconnected metrics on $A \in \exp X$.

$\mathcal{UDM} = \bigcup\{\mathcal{UDM}(A) : A \in \exp X,\ |A| \geq 2\}$

**Theorem 4.** *Let* $(X,d)$ *be a compact ultrametric space. Then there exists a map* $w\colon \mathcal{UDM} \to \mathcal{UDM}(X)$ *that satisfies analogous properties to those listed in Theorem 3.*