Research Statement

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1 Introduction

My research is in topology and its border with analysis. One important topic of my scientific interests is the map extension theory, in particular, simultaneous extension of families of real-valued functions, (pseudo)metrics and special maps defined on subspaces of topological spaces. I work on problems of existence and construction of continuous extension operators that preserve properties, operations and structures given on sets of partial maps. Problems in this field trace back to the classical Tietze-Urysohn extension theorem [46] for real functions, the result of Hausdorff [17] on extending a fixed metric from a closed subspace of a metrizable space, and their generalizations. My other research topics include continuum theory, aspects of geometric measure theory such as spaces of finite and σ-finite Hausdorff measure, the theory of continuous selection for multi-valued maps, special metrics and hyperspace theory.

In the following sections of this statement I discuss background and results of my doctoral and postdoctoral research as well as current projects and my plans for future work.

2 Background in Map Extension

There is a long history of improvements to the Tietze-Urysohn extension theorem. J. Dugundji [11] proved that if A is a closed subset of a metric space then there is a continuous (with respect to the topologies of pointwise and uniform convergence), linear, regular extension operator from $C(A)$, the space of continuous real-valued functions on A, to the space $C(X)$. H.P. Kunzi and L. Shapiro [19] improved the extension theorem of Dugundji for functions with compact variable domains as follows:

Let (X, d) be a metric space and $C_{vc} = \{f \colon A \to \mathbb{R} \mid A \subset X \text{ is compact and } f \text{ is continuous }\}$. Then C_{vc} is a metric space where the distance between two functions f and g is given by the Hausdorff distance between their graphs which are closed, bounded subsets of $X \times \mathbb{R}$. Let $C^*(X)$ denote the set of continuous, bounded, real-valued functions on X.

Theorem (H.P. Kunzi, L. Shapiro [19])**.** *Suppose that X is a metrizable space and that the set $C^*(X)$ is endowed with the topology of uniform convergence. Then there exists a continuous, regular, linear extension operator $\Phi \colon C_{vc} \to C^*(X)$.*

Note that since all the functions are defined on compact sets, they are automatically uniformly continuous. It is not proved that the Kunzi-Shapiro operator preserves uniformly continuous functions if X is not compact. It becomes natural to ask for variants of this result for continuous and uniformly continuous functions defined on closed subsets of the space X with no compactness assumption.

The theory of extensions of metrics develops in parallel with the theory of extensions of functions. In the 90's C. Bessaga [7] raised a question of existence of continuous linear operators extending the cone of metrics defined on a fixed closed subset of a metrizable space and solved it for some special cases. T.Banakh [3] was first to obtain the complete solution of the mentioned problem which is an analogue of the Dugundji theorem for the case of metrics.

A new direction of research in this area is related to construction of operators simultaneously extending metrics with variable domains. Generalizations of Banakh's and Bessaga's theorems were obtained by E.D. Tymchatyn and M. Zarichnyi [47] for the restricted case of a compact space.

For a metric space (X, d) let $\exp_b(X)$ be the family of bounded, closed and non-empty subsets of X. Denote by $\mathcal{PM}(A)$ and $\mathcal{M}(A)$ the families of bounded, continuous pseudometrics and metrics, respectively, defined on A for every $A \in \exp_b(X)$. Then the sets $\mathcal{PM} = \bigcup\{\mathcal{PM}(A) \mid A \in \exp_b(X)\}$ and $\mathcal{M} = \bigcup\{\mathcal{M}(A) \mid A \in \exp_b(X),\ |A| \geq 2\}$ can be viewed as subspaces of the metric space $\exp_b(X \times X \times \mathbb{R})$ endowed with the Hausdorff metric under the identification of every partial (pseudo)metric with its graph $\rho \to \Gamma_\rho$.

Definition. *An extension operator* $u\colon \mathcal{PM} \to \mathcal{PM}(X)$ *is a map such that for any* $\rho \in \mathcal{PM}$, $u(\rho)$ *is a continuous pseudometric on* X *with* $u(\rho)|_{\mathrm{dom}\rho \times \mathrm{dom}\rho} = \rho$.

We say u *is linear if* $u(\alpha\rho + \beta\tau) = \alpha u(\rho) + \beta u(\tau)$ *for* $\alpha, \beta \geq 0$ *and* $\rho, \tau \in \mathcal{PM}(A)$ *for* $A \in \exp_b(X)$.

For $\rho \in \mathcal{PM}$ *let* $\|\rho\| = \sup\{\rho(x, y) \mid x, y \in \mathrm{dom}\rho\}$. *We say* u *is regular if* $\|u(\rho)\| = \|\rho\|$ *for each* $\rho \in \mathcal{PM}$.

Theorem (E.D. Tymchatyn, M. Zarichnyi [47])**.** *Let* X *be a compact metrizable space. There exists a continuous, linear, regular extension operator* $u\colon \mathcal{PM} \to \mathcal{PM}(X)$.

The same authors in [48] considered a similar problem of simultaneous extension of partial ultrametrics.

Definition. *A metric* r *on a set* Y *is an ultrametric or a non-Archimedean metric if* $r(x, y) \leq \max\{r(x, z), r(y, z)\}$ *for all* $x, y, z \in Y$.

It is known [16] that a metric space (Y, r) admits an ultrametric compatible with its topology if and only if $\dim Y = 0$. For a zero-dimensional metric space (X, d) and any $A \in \exp_b(X)$ let $\mathcal{UM}(A)$ denote the family of continuous, bounded ultrametrics on A. Equip the set $\mathcal{UM} = \bigcup\{\mathcal{UM}(A) \mid A \in \exp_b(X),\ |A| \geq 2\}$ with the Hausdorff metric topology. Note that the sum of two ultrametrics is not in general an ultrametric but the maximum is.

Theorem (E.D. Tymchatyn, M. Zarichnyi [48])**.** *Let* (X, d) *be a zero-dimensional metric compactum. There exists a regular, continuous extension operator* $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ *such that* $u(\max\{\rho, \rho'\}) = \max\{u(\rho), u(\rho')\}$ *for* $\rho, \rho' \in \mathcal{UM}$ *with a common domain.*

3 Extension of Metrics and Functions

One may ask for ways to strengthen the Tymchatyn-Zarichnyi results by omitting the compactness hypothesis for the underlying space X and/or changing the hyperspace topology. Another group of interesting questions arising naturally in the context of the mentioned results is related to existence of extension operators for other classes of special metrics such as convex, Lipschitz, fuzzy metrics, metrics with the four-point property, etc.

In my Ph.D. dissertation "Extensions of Metric Structures" I obtained several improvements and analogues of the Tymchatyn-Zarichnyi theorems mentioned above. I considered problems on extensions of (pseudo)metrics with special properties, in particular ultrametrics, uniformly disconnected metrics and semicontinuous (pseudo)metrics.

3.1 Extending semicontinuous metrics

In [44] I considered problems of extensions of upper semicontinuous metrics defined on convex subsets of locally convex spaces. In order to topologize the set of partial upper semicontinuous metrics I used the Fell topology and the identification of each metric with its hypograph which is a closed set in the corresponding space (see [6]).

Theorem 1 (I. Stasyuk [44]). *Let X be a convex, metrizable, compact subspace of a locally convex space and let A be a closed subset of X. Then there exists an operator extending upper semicontinuous metrics from A to X. This operator preserves convex combinations of metrics and is continuous with respect to the Fell topology.*

3.2 Extending uniformly disconnected metrics and (Lipschitz) ultrametrics

Identifying partial maps with their graphs or subgraphs I constructed extension operators that are continuous in various hyperspace topologies. Moreover, these operators preserve operations defined for partial (pseudo)metrics.

In [42] and [45] I constructed extension operators for ultrametrics that not only possess all the properties of the Tymchatyn-Zarichnyi operator from [48] but are also positive homogeneous and continuous with respect to the pointwise convergence topology. These results rely essentially on the zero-dimensional Michael selection theorem for multi-valued maps [24]. In order to ensure that the extension of an individual metric is actually a metric (not merely a pseudometric that allows zero distance between distinct points), I represented my extension operator in terms of a countable family of extension operators that preserve pseudometrics and separate points. I have been frequently applying modifications of this technique in my other projects on extensions of metrics. It turned out that a similar outcome could be obtained for a wider class of metrics, namely uniformly disconnected metrics.

Definition. *A metric space (Y,ρ) is called c-uniformly disconnected, if there exists a constant $c>0$ such that*

$$c\rho(y_0,y_n)\le \max\{\rho(y_{j-1},y_j)\mid j\in\{1,\dots,n\}\}$$

for every finite chain $y_0,y_1,\dots,y_n$ of elements of the set Y. The metric ρ is called uniformly disconnected.

These metrics are precisely the bi-Lipschitz images of ultrametrics. For a compact, zero-dimensional metrizable space X and its closed subset A let $\mathcal{UDM}(A)$ be the set of all continuous, uniformly disconnected metrics on A and let $\mathcal{UDM}=\bigcup\{\mathcal{UDM}(A)\mid A\in\exp_b(X),\ |A|\ge 2\}$ be equipped with the Hausdorff metric topology.

Theorem 2 (I. Stasyuk [41]). *Let X be a compact, zero-dimensional, metrizable space. There exists a map $v\colon\mathcal{UDM}\to\mathcal{UDM}(X)$ with the following properties for all $\sigma,\sigma'\in\mathcal{UDM}$:*

1) v is continuous;

2) the restriction $v|_{\mathcal{UDM}(A)}\colon\mathcal{UDM}(A)\to\mathcal{UDM}(X)$ is continuous with respect to the topology of pointwise convergence on $\mathcal{UDM}(A)$ and $\mathcal{UDM}(X)$ for every $A\in\exp_b(X)$ with $|A|\ge 2$;

3) $v(\sigma)$ is an extension of σ over X;

4) $\|v(\sigma)\| = \|\sigma\|$*;*

5) $v(\max\{\sigma, \sigma'\}) = \max\{v(\sigma), v(\sigma')\}$ *if the domains of* σ *and* σ' *are the same;*

6) $v(c\sigma) = cv(\sigma)$ *for every constant* $c > 0$*;*

7) if σ *takes only binary rational values then so does* $v(\sigma)$*;*

8) if σ *is an ultrametric so is* $v(\sigma)$*.*

9) $\dim_A(X, v(\sigma)) = \dim_A(\mathrm{dom}\sigma, \sigma)$*;*

Here $\dim_A$ denotes the Assouad dimension, an important bi-Lipschitz invariant of a metric space (see for instance [21]).

The main result of [43] was my first successful attempt in proving the existence of continuous extension operators for partial ultrametrics with no requirement of compactness of the initial space.

Theorem 3 (I. Stasyuk [43])**.** *Let* X *be a zero-dimensional, separable, metrizable topological space and let* A *be any closed subset of* X *with* $|A| \geq 2$*. There exists a positive homogeneous, operator* $v\colon \mathcal{UM}(A) \to \mathcal{UM}(X)$ *preserving maxima of ultrametics. This operator is continuous with respect to the pointwise convergence topology.*

In my postdoctoral work I have obtained a series of general theorems on extensions of real-valued functions and metrics in collaboration with E.D. Tymchatyn and other coauthors. In [37] we obtained the strongest result on simultaneous extensions of ultrametrics so far.

Theorem 4 (I. Stasyuk, E.D. Tymchatyn [37])**.** *Let* (X, d) *be a bounded complete ultrametric space. There exists an operator* $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ *that satisfies the following conditions for every* $\rho, \rho_1 \in \mathcal{UM}$ *and* $c > 0$*:*

1) $u(\rho)$ *is an extension of* ρ *over* X*;*

2) u *is positive-homogeneous i.e.* $u(c\rho) = cu(\rho)$*;*

3) $u(\max\{\rho, \rho_1\}) = \max\{u(\rho), u(\rho_1)\}$ *if* $\mathrm{dom}\rho = \mathrm{dom}\rho_1$*;*

4) $\|u(\rho)\| = \|\rho\|$*;*

5) *If* $\{\rho_n\}$ *is a sequence in* $\mathcal{UM}$ *such that* $\{\Gamma_{\rho_n}\}$ *converges to* Γ_ρ *for some* $\rho \in \mathcal{UM}$ *in the Hausdorff metric then* $\{u(\rho_n)\}$ *converges to* $u(\rho)$ *pointwise on* $X \times X$*;*

In order to construct the operator we proved an auxiliary result which is a "uniformly continuous" analogue of a selection theorem due to Choban [10]. We also proved a counterpart of the above result for uniformly continuous ultrametrics.

In the joint work with T. Banakh, N. Brodskiy and E.D. Tymchatyn [5] we obtained a similar theorem for the classes of uniformly continuous ultrametrics and metrics. One of the remarkable properties of the constructed operator is that it preserves Lipschitz (ultra)metrics.

3.3 Extension operators for convex metrics

Another class of special metrics for which we have constructed extensions is the class of convex metrics.

Definition. *A Peano continuum is a compact, connected and locally connected metric space.*

Definition. *A metric r on a Peano continuum X is said to be convex if for each $x, y \in X$ there is an arc $[xy]$ with endpoints x and y such that $[xy]$ is isometric to the closed interval $[0, r(x, y)]$ in the real line $\mathbb{R}$.*

It is known (Bing [8]) that a metric continuum is locally connected if and only if it admits a convex metric.

Theorem (R.H. Bing [8])**.** *Let X be a Peano continuum and let A be a locally connected subcontinuum of X. Then any continuous convex metric on A extends to a continuous convex metric on X.*

If X is a Peano continuum and A is a locally connected subcontinuum of X, let $\mathcal{CM}(A)$ denote the set of continuous convex metrics on A. Let

$$\mathcal{CM} = \bigcup\{\mathcal{CM}(A) \mid A \text{ is a Peano subcontinuum of } X\}$$

be the set of partial convex metrics endowed with the Hausdorff distance. Since the sum of two convex metrics is not necessary convex, one cannot talk about linearity of an extension operator. Improving Bing's technique by choosing in a canonical way a modulus function for each member of $\mathcal{CM}$ we obtained a generalization of his result for the case of variable domains:

Theorem 5 (I. Stasyuk, E.D. Tymchatyn [36])**.** *Let X be a Peano continuum. There exists a continuous extension operator $u\colon \mathcal{CM} \to \mathcal{CM}(X)$.*

3.4 Extending pairs of metrics

In [35] we considered a variation of the problem on extending a metric that involves extensions of pairs of metrics defined on non-disjoint subspaces of a topological space. For a metric space X let

$$\mathcal{P} = \{(\rho_1, \rho_2) \in \mathcal{M}(A) \times \mathcal{M}(B) \mid \ A, B \in \exp_b(X),\ A \cap B \neq \varnothing, |A| \geq 2, |B| \geq 2, \\ \rho_1 = \rho_2 \text{ on } (A \cap B) \times (A \cap B)\}.$$

The set $\mathcal{P}$ consists of all pairs of partial, continuous, bounded metrics which agree on the non-empty intersection of their domains. Two pairs of metrics are close if their corresponding graphs are close and if the intersections of their domains are close in the Hausdorff metric. Let $\mathcal{M}$ stand for the set of all continuous, bounded metrics with closed and bounded domains in X.

The following theorem answers in the negative a question of T.Banakh and M.Zarichnyi:

Theorem 6 (I. Stasyuk, E.D. Tymchatyn [35])**.** *There exists no linear or even subadditive extension operator from $\mathcal{P}$ to $\mathcal{M}$.*

In addition we obtained the following results:

Theorem 7 (I. Stasyuk, E.D. Tymchatyn [35])**.** *There exists a positive homogeneous extension operator $u\colon \mathcal{P} \to \mathcal{M}$ with the following properties:*

(i) $u((\rho_1, \rho_2) + (\sigma_1, \sigma_2)) \geq u(\rho_1, \rho_2) + u(\sigma_1, \sigma_2)$ *for all* $(\rho_1, \rho_2), (\sigma_1, \sigma_2) \in \mathcal{P}$ *with* $\mathrm{dom}\rho_1 = \mathrm{dom}\sigma_1$ *and* $\mathrm{dom}\rho_2 = \mathrm{dom}\sigma_2$.

(ii) the restriction of u *onto the set of pairs of uniformly continuous metrics is continuous.*

Theorem 8 (I. Stasyuk, E.D. Tymchatyn [35])**.**

1) *There is no linear extension operator preserving pairs of Lipschitz metrics.*

2) *There is no extension operator for pairs of Lipschitz ultrametrics that preserves the operation maximum.*

3.5 Extension operator for metrics in the non-compact case

Quite recently we were able to get a rather general version of the theorem of E.D. Tymchatyn and M. Zarichnyi [47]) for the case of a noncompact metric space. To extend methods used in [47] we needed a non-compact version of a selection theorem due to Fryszkowski [14] used in [47]. Applying the Ageev-Repovs selection theorem [1] we proved the following statement:

Theorem 9 (I. Stasyuk, E.D. Tymchatyn [31])**.** *Let* (X, d) *be a complete metric space. There is a linear, regular extension operator* $w\colon \mathcal{PM} \to \mathcal{PM}(X)$ *which is continuous with respect to the topology of uniform convergence on compact sets on* $\mathcal{PM}(X)$*.*

In order to get an extension that would preserve metrics we could not apply the method of separating points of X by a countable family of operators because of possible nonseparability of X. Instead we represented our extension operator as the sum of the operator from Theorem 9 and another special operator for pseudometrics which, unfortunately, increased the norm making the obtained operator only "almost regular". Along with the Repovs-Ageev selection theorem we applied properties of certain Milyutin maps which are maps between topological spaces admitting continuous functions from their range to the set of probability measures on their domains (see for instance, [2]). The following is an analogue of Theorem 9 for the case of metrics.

Theorem 10 (I. Stasyuk, E.D. Tymchatyn [31])**.** *Let* (X, d) *be a complete metric space. For arbitrary* $\eta > 0$ *there exists a linear extension operator* $u\colon \mathcal{M} \to \mathcal{M}(X)$ *which is continuous with respect to the topology of uniform convergence on compact sets on* $\mathcal{M}(X)$ *and* $\|u(\rho)\| \leq (1+\eta)\|\rho\|$ *for every* $\rho \in \mathcal{M}$*.*

It seems reasonable to expect nonexistence of extension operators possessing the entire set of properties of the Tycmhatyn-Zarichnyi construction from Theorem 2 for rather general settings. Currently I am trying to prove that our Theorem 10 is sharp.

Conjecture. *There exists no regular extension operator for metrics that satisfies all the properties of the operator from Theorem 10.*

3.6 Extensions of real-valued functions

One of the problems of particular interest to me are related to finding improvements of the Kunzi-Shapiro theorem on simultaneous extensions of real-valued functions. Its proof seems to depend essentially on the compactness of domains of partial functions. In my joint work with A.Koyama, E.D. Tymchatyn and A.Zagorodnyuk [18] we obtained a variant of this result for functions with closed and bounded domains in a complete metric space. Recall that for a metric space X we let $\exp_b(X)$ be the space of non-empty, closed and bounded subsets of X with the Hausdorff metric. For $A \in \exp_b(X)$ let $C^*(A)$ denote the family of continuous, bounded, real-valued functions on A. Let $C_b^* = \bigcup\{C^*(A) \mid A \in \exp_b(X)\}$.

Theorem 11 (A. Koyama, I. Stasyuk, E.D. Tymchatyn, A. Zagorodnyuk [18])**.** *Let (X, d) be a complete metric space. There exists a regular, linear extension operator $e\colon C_b^* \to C^*(X)$. This operator is continuous with respect to the topology of uniform convergence on compact sets on $C^*(X)$.*

Our proof uses Milyutin maps and Michael's zero-dimensional selection theorem and is much simpler than that of Theorem (H.P. Kunzi, L. Shapiro [19]).

In [5], improving McShane's technique [23], we obtained an analogue of the Kunzi-Shapiro theorem for the class of uniformly continuous functions with closed domains. Let (X, d) be a bounded metric space and let $\exp(X)$ stand for the set of closed non-empty subsets of X. For every $A \in \exp(X)$ we denote by $C_u^*(A)$ the set of uniformly continuous and bounded real-valued functions on A and let $C_u^* = \bigcup\{C_u^*(A) \mid A \in \exp(X)\}$. As before we topologize the set of partial functions by identifying each function with its graph and equipping it with the Hausdorff metric.

Theorem 12 (T. Banakh, N. Brodskiy, I. Stasyuk, E.D. Tymchatyn [5])**.** *There exists a continuous, regular, positive homogeneous extension operator $v\colon C_u^* \to C_u^*(X)$ that preserves Lipschitz functions and Lipschitz constants.*

It is known that in general there is no analogue of the Dugundji theorem on linear extensions of continuous, real functions defined on a closed subset of a metric space for the case of uniformly continuous functions (see [28, Remarks to §2]). So our operator cannot be improved to become linear.

4 Selections for Multi-Valued Maps

In my research on extension of functions and metrics I rely essentially on methods from the theory of continuous selections for multi-valued maps. Special selection theorems have been obtained as auxiliary tools in my work in the extension theory. Recall that if F is a multi-valued map from a space X to a space Y then $f\colon X \to Y$ is single-valued selection for F if for every $x \in X$, $f(x) \in F(x)$. In [33] we study conditions for existence of selections with stronger properties than being merely continuous. If X is a zero-dimensional space and Y is a complete metric space it is known that one cannot always choose a selection of a multivalued map $F\colon X \to Y$ to be uniformly continuous even if F is uniformly continuous with respect to the Hausdorff distance. The main theorem of [33] is as follows:

Theorem 13 (I. Stasyuk, E.D. Tymchatyn [33])**.** *Let (X, d) be an ultrametric space and (Y, ρ) be a complete metric space. Then every uniformly continuous with respect to Hausdorff distance multi-valued map $F\colon (X, d) \to (Y, \rho)$ has a uniformly continuous selection.*

The proof relies on properties of non-Archimedean distances and I am planning to look for ways to extend it for a wider class of metric spaces.

5 Spaces of Finite Hausdorff Measure

Another subject of interest in my research is related to properties of spaces that admit finite and σ-finite n-dimensional Hasudorff measure $\mathcal{H}^n$ where n is a positive integer. These spaces are important generalizations of n-dimensional polyhedra. Let us recall several basic concepts.

Definition. *Let (X, ρ) be a separable metric space and $\alpha \geq 0$. Then the α-dimensional Hausdorff measure $\mathcal{H}^\alpha_\rho$ on X is defined by*

$$\mathcal{H}^\alpha_\rho(A) = \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{diam}_\rho(U_i))^\alpha \mid A \subset \bigcup_{i=1}^{\infty} U_i \subseteq X, \mathrm{diam}_\rho(U_i) < \delta \textit{ for every } i \in \mathbb{N} \right\}$$

for any $A \subset X$. We call $\mathcal{H}^1_\rho$ the linear Hausdorff measure on (X, ρ).

Definition. *The n-dimensional Nöbeling space ν^{2n+1}_n is the subspace of the Euclidean space $\mathbb{R}^{2n+1}$ which consists of all points with at most n rational coordinates.*

The space ν^{2n+1}_n is universal for separable metric spaces of dimension at most n.

Characterization theorems for spaces admitting finite 1-dimensional (or linear) Hausdorff measure were obtained by Eilenberg and Harrold in the 1930's [12]. It is well-known that the property of having finite linear measure is not preserved under finite unions of closed sets. Mauldin proved that if X is a compact metric space which is the union of finitely many closed sets each of which admits a σ-finite linear measure then X admits a σ-finite linear measure. In [32] we answered in the strongest possible way a 1989 question of Mauldin:

Theorem 14 (I. Stasyuk, E.D. Tymchatyn [32])**.** *Let $X = \bigcup_{i=1}^{\infty} X_i$ be a topological space where each subspace X_i is totally regular and closed in X. Then the space X can be embedded in the 1-dimensional N´obeling space $\nu^3_1 \subset \mathbb{R}^3$ so that the image of X has σ-finite linear measure with respect to the usual metric on ν^3_1.*

6 Existence of Convex Metrics on Non-Compact Spaces

Definition. *Let (X, ρ) be a metric space. We say that ρ is convex if for each $x \neq y \in X$ there is an arc $A \subset X$ with end-points x and y such that $(A, \rho|_{A\times A})$ is isometric to the interval $[0, \rho(x, y)]$ in the real line.*

In the 1950's Bing, [8], proved that every Peano continuum (i.e., a compact, connected and locally connected metric space) admits a compatible convex metric. He asked for an extension of his theorem to non-compact spaces. In [27] we proved that such an extension of Bing's result is possible for a quite large class of metric spaces.

Our approach is a modification of that of Bing. Bing constructed in a Peano continuum X a special decreasing sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of finite partitions of X. Then he assigned weights $w_n(U)$ to the elements U of $\mathcal{U}_n$. A subcollection of $\mathcal{U}_n$ had weight equal to the sum of the weights of its elements. An approximation to the distance $\rho(x, y)$ between two points x and y of X was given by the smallest of the weights $w_n(\mathcal{C}_n)$ of chains $\mathcal{C}_n$ in $\mathcal{U}_n$ from x to y. He showed that $\limsup \bigcup \mathcal{C}_n$ contains a line segment (with respect to ρ) from x to y.

In the non-compact case, $\limsup \bigcup \mathcal{C}_n$ need not in general contain a connected subset from x to y. Consider for instance $X = [0, 1] \times [0, 1] - (0, 1) \times \{0\}$ in its usual metric inherited from the plane and choose $x = (0,0)$ and $y = (1,0)$. Hence, in the non-compact case, extra care needs to be taken in assigning weights to the elements of $\mathcal{U}_n$ to ensure that in the above $\limsup \bigcup \mathcal{C}_n$ is a continuuum containing x and y. We do this by defining finite connected graphs $G_1 \subset G_2 \subset \dots$ in X so that G_i is similar to the 1-skeleton of the nerve of $\mathcal{U}_i$. We then assign weights to the elements of $\mathcal{U}_i$ so that the elements of $\mathcal{U}_{i+1}$ which meet G_i are relatively light and light chains in $\mathcal{U}_{i+1}$ between distant points of X lie (except possibly near their ends) along G_i. We restrict our attention to spaces admitting sufficient finite partitions.

Definition. *A metric space (X, ρ) has property S if for each $\varepsilon > 0$ there is a finite cover of X by connected sets of diameter less than ε.*

Note that $\mathbb{R}$ in its usual metric does not have property S while $(0,1)$ in its usual metric does. So this property is a metric property. If (X, d) has property S, then it is locally connected and totally bounded. It is also known that every locally connected metric continuum has property S. The following theorem is the main result of [27].

Theorem 15 (J. Nikiel, I. Stasyuk, M. Tuncali, E.D. Tymchatyn [27])**.** *If X is a connected and locally arc-connected metric space with property S, then X admits a convex metric.*

7 Future Plans

Currently I have several ongoing projects. One is devoted to the characterization of spaces of n-dimensional finite measure in collaboration with M. Tuncali and E.D. Tymchatyn. Another project is on convexification problems inspired by our recent result on existence of convex metrics on non-compact spaces. I am also working on a survey of results on extension of metrics together with T. Banakh, E.D. Tymchatyn and M. Zarichnyi.

7.1 Extension of maps

One set of questions that I intend to work on has been inspired by recent work of D. Repovš and M. Zarichnyi [29] on linear extensions of Lipschitz (pseudo)metrics. Let A be a closed subset of a compact metric space (X, d). Denote by $\mathcal{PM}_{\mathrm{Lip}}(A)$ (resp. $\mathcal{M}_{\mathrm{Lip}}(A)$) the sets of all Lipschitz pseudometrics (resp. metrics) on A. Equip the set $\mathcal{PM}_{\mathrm{Lip}}(A)$ with the seminorm defined as follows:

$$\|\rho\|_A = \sup \left\{ \frac{\rho(x,y)}{d(x,y)} \mid x, y \in A,\ x \neq y \right\}.$$

An operator $u \colon \mathcal{PM}_{\mathrm{Lip}}(A) \to \mathcal{PM}_{\mathrm{Lip}}(X)$ is said to be continuous if $\|u\| = \sup\{\|u(\rho)\|_X \mid \|\rho\|_A \leq 1\}$ is finite. It is proved in [29] that there exists a closed subspace of a zero-dimensional compact metric space admitting no continuous linear extension operator from $\mathcal{PM}_{\mathrm{Lip}}(A)$ to $\mathcal{PM}_{\mathrm{Lip}}(X)$. It is natural to ask questions on existence of linear extension operators for Lipschitz (pseudo)metrics that would be continuous in other topologies. Given a compact metric space X let

$$\mathcal{PM}_{\mathrm{Lip}} = \bigcup \{\mathcal{PM}_{\mathrm{Lip}}(A) \mid A \neq \varnothing \text{ is closed in } X\}.$$

Problem. Equip the set $\mathcal{PM}_{\mathrm{Lip}}$ with the metric D:

$$D(\rho_1, \rho_2) = \inf\{\|\tilde{\rho}_1 - \tilde{\rho}_2\| \mid \tilde{\rho}_i \text{ is a Lipschitz extension of } \rho_i\}.$$

Is there a continuous linear extension operator for partial Lipschitz pseudometrics which is continuous with respect to the metric D?

Problem. The space $\mathcal{PM}_{\mathrm{Lip}}$ can be endowed with the topology of uniform convergence. Are there linear extension operators from $\mathcal{PM}_{\mathrm{Lip}}$ to $\mathcal{PM}_{\mathrm{Lip}}(X)$ that are continuous in this topology?

Problem. Is there a continuous extension operator for Lipschitz ultrametrics preserving the operation max?

A positive answer to the last question would improve our operator from [5] which preserves Lipschitz ultrametrics and is positive homogeneous.

Our joint work [34] with E.D. Tymchatyn deals with the problem of simultaneous extension of fuzzy ultrametrics defined on closed subsets of a complete fuzzy ultrametric space. These structures are analogues of Menger probabilistic metric structures and have been first introduced quite recently (see for instance [15], [26], [30]). We constructed a continuous extension operator that preserves the operation of pointwise minimum of fuzzy ultrametrics with common domain and an operation which is an analogue of multiplication by a constant defined for fuzzy ultrametrics. A similar problem can be posed for fuzzy metrics.

Problem. Construct extension operators for fuzzy metrics defined on closed subsets of a given fuzzy metric space that preserve operations defined for partial fuzzy metrics with common domain.

Recently Taras Banakh suggested to me the following question on possible combining of properties of extension operators for metrics and ultrametrics:

Problem. Let X be a zero-dimensional metric space. Does there exist a linear operator simultaneously extending continuous metrics from closed subsets of X which preserves ultrametrics and the operation maximum for ultrametrics with common domains?

It seems that the answer to this question has to be negative and counterexamples that we constructed in [35] should help to understand possible approaches to prove this.

Moreover, I intend to work on generalizing my results on simultaneous extensions of functions transferring them to a wider class of maps that are not necessarily real-valued.

Problem. What counterparts of the extension theorems from [5] and [18] can be proved for functions with values in Banach spaces and general metric spaces?

Techniques that can be applied to approach these questions in particular would involve integration of functions with values in Banach spaces.

7.2 Characterization of spaces of finite Hausdorff measure

R. Buskirk, J. Nikiel and E.D. Tymchatyn showed in [9] that every space admitting finite linear Hausdorff measure is homeomorphic to the inverse limit of a sequence of connected graphs with monotone bonding maps. The problem of generalizing this result for higher dimensions is of particular interest and currently I work on it in collaboration with M. Tuncali and E.D. Tymchatyn. One of the results that we are expecting to obtain is to express the underlying space as an inverse limit of n-dimensional polyhedra with a finite upper bound for the measures of the polyhedra (as a special version of the Freudenthal theorem). We also require that the sequence of polyhedra are convergent to a homeomorphic copy of the initial space in the Hausdorff distance. A possible approach is to find a Lipschitz embedding of our space into a specific Menger cube of Assouad dimension n described in the paper of Luukkainen [21]. The idea is to find such a homeomorphism using classical embedding results into the n-dimensional Nöbeling space and then reembed the space into the Menger cube. For this purpose we are using the recent result of Le Donne [20] stating that any compact metric space of Hausdorff dimension n can be embedded in $\mathbb{R}^k$ via a Lipschitz map where $k = 2n + 1$. Another possible approach is to use net measures rather than Hausdorff measures. In the definition of a net measure the class of sets that give approximations to the measure of the space are disjoint cubes (in our case in $\mathbb{R}^{2n+1}$). Note that an n-dimensional space has finite n-dimensional Hausdorff measure if and only of it has a finite n-dimensional net measure.

We have already obtained positive results for certain special cases. If a general characterization could be obtained, one would pose questions related to describing properties of spaces having σ-finite Hausdorff measure for higher dimensions.

7.3 Convexification problems

Currently I am working with E.D. Tymchatyn and M. Tuncali on possible generalizations of our result from [27] on existence of convex metrics to wider classes of spaces.

Note that property S is a sufficient condition for a space to have a decreasing sequence of finite partitions (see [8]).

Problem. Suppose that X is a connected and locally arc-connected, separable, metric space which admits locally finite covers by small open and connected sets. Does X admit a convex metric generating its topology? Is property S in Theorem 15 necessary?

Problem. Can Theorem 15 be extended to the case of spaces admitting infinite partitions?

It is quite reasonable to expect that the idea of assigning weights to elements of partitions could work for the case of nonfinite partitions as well.

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