ON CONTINUOUS EXTENSION OF UNIFORMLY CONTINUOUS FUNCTIONS AND METRICS

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Abstract. We prove that there exists a continuous regular, positive homogeneous extension operator for the family of all uniformly continuous bounded real-valued functions whose domains are closed subsets of a bounded metric space (X, d). In particular, this operator preserves Lipschitz functions. A similar result is obtained for partial metrics and ultrametrics.

1. Introduction

The theory of extensions of continuous functions and (pseudo)metrics has a long history. The Tietze-Urysohn theorem asserts that every continuous real-valued function on a closed subset of a metric space X admits a continuous extension to X. Hausdorff [7] proved an analogous theorem for metrics. McShane [9] showed that every uniformly continuous real-valued function from a closed subset of a metric space X which admits a concave modulus function φ such that $\lim_{t\to 0}\varphi(t) = 0$ has a uniformly continuous extension to X. It is easy to see that the above condition on the modulus function is necessary. The chief contribution of this paper is to show that McShane's technique may be modified to give a continuous extension operator for several classes of functions. We also use a modification of Bing's formula [4] to construct continuous operators extending uniformly continuous metrics and ultrametics defined on the family of closed subsets of a metric (ultrametric) space.

Dugundji [5] proved that one could extend continuously all functions with a fixed domain.

Theorem 1.1 (Dugundji, 1951). *Let X be a metric space and A its closed subset. Let $C^*(A)$ denote the space of all bounded real-valued continuous functions with supnorm metric. There exists a continuous, regular(of unit norm), linear extension operator $\Phi\colon C^*(A) \to C^*(X)$.*

The question of existence of linear operators extending cones of (pseudo)metrics was raised and solved for some special cases by C. Bessaga [3]. T. Banakh [1] was first to obtain a complete solution of this problem. Linear extension operators preserving metrics were also constructed by O. Pikhurko [10] and M. Zarichnyi [15]. We may summarize results obtained by these authors as follows.

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Theorem 1.2 (C.Bessaga, 1993, T.Banakh, 1994, O. Pikhurko, 1994, M. Zarichnyi, 1996)**.** *Let (X,d) be a metric space and A a closed subset of X. There exists a continuous, regular, linear extension operator from the set of continuous (pseudo)metrics on A to the set of continuous (pseudo)metrics on X.*

Stepanova [12] first considered simultaneous extension of functions with variable domains.

Theorem 1.3 (Stepanova, 1993)**.** *Let (X,d) be a metric space. There exists a continuous extension operator from the space of real-valued functions whose domains are compact subsets of X to $C^*(X)$.*

Stepanova showed that metrizability of X was a necessary condition.

In 1997 Künzi and Shapiro [8] improved Stepanova's result by obtaining an extension operator which was also linear. In 2004 Tymchatyn and Zarichnyi [13] obtained a weak version of the Künzi-Shapiro result for pseudometrics.

Theorem 1.4 (Tymchatyn, Zarichnyi, 2004)**.** *Let (X,d) be a metric compactum. There exists a regular, linear, continuous with respect to the uniform topology operator extending continuous pseudometrics defined on closed subsets of X.*

In this paper we drop the compactness assumptions in Theorems 1.3 and 1.4 and obtain some partial generalizations of those theorems on simultaneous extension operators for functions and metrics defined on closed subsets of a bounded metric space.

2. Preliminaries

A function $\varphi\colon [0,\infty)\to[0,\infty)$ is said to be *concave* if φ is continuous and the set $\{(x,t) : 0\le x \text{ and } 0\le t\le\varphi(x)\}$ is a convex subset of $\mathbb{R}\times\mathbb{R}$. Let

$$\mathcal{J}=\{\varphi\in C([0,\infty)) : \varphi \text{ is bounded, concave, non-decreasing and } \varphi(0)=0\}.$$

Note that each $\varphi\in\mathcal{J}$ is uniformly continuous because it is both non-decreasing and bounded. Note also that $\mathcal{J}$ is closed in the space $C^*_u([0,\infty))$ of bounded uniformly continuous real-valued functions with supnorm metric. Also note that if $\varphi\in\mathcal{J}$ and $a,b\ge 0$ then $\varphi(a+b)\le\varphi(a)+\varphi(b)$.

For the remainder of this paper (X,d) will be a bounded metric space (in Section 5 we will assume that d is an ultrametric). If $A\subset X$, $f\colon A\to\mathbb{R}$ is a function and $\varphi\in\mathcal{J}$ so that $|f(x)-f(y)|\le\varphi(d(x,y))$ for all $x,y\in A$ then we call φ a *modulus function for f*. Similarly, if ρ is a metric on A and $\varphi\in\mathcal{J}$ so that $\rho(x,y)\le\varphi(d(x,y))$ for all $x,y\in A$ then we call φ a *modulus function for ρ*. Note that we require a modulus function to be bounded, concave, non-decreasing with fixed point 0.

Let $\exp(X)$ stand for the set of closed non-empty subsets of X. For every $A\in\exp(X)$ let $C^*_u(A)$ be the set of uniformly continuous and bounded real-valued functions on A. We will write $\mathrm{dom} f=A$ if $f\in C^*_u(A)$, $A\in\exp(X)$. Let

$$C^*_u=\bigcup\{C^*_u(A)\mid A\in\exp(X)\}.$$

Assume that each $f\in C^*_u$ is identified with its graph $\Gamma_f=\{(x,f(x)) : x\in\mathrm{dom} f\}$ which is a bounded and closed subset of $X\times\mathbb{R}$. Let $\tilde d$ be the l_1 metric on $X\times\mathbb{R}$ given by the formula

$$\tilde d[(x,t),(x',t')]=d(x,x')+|t-t'|.$$

For $f, g \in C_u^*$ define $H(f,g)$ to be the Hausdorff distance between Γ_f and Γ_g induced by $\tilde{d}$. Let $\|f\|$ denote the supnorm of f that is $\|f\| = \sup\{|f(x)| : x \in \mathrm{dom} f\}$. For any set M in the real plane $\mathbb{R} \times \mathbb{R}$ let $co(M)$ denote the closed convex hull of M.

The next proposition was essentially proved in [6, page 116].

Proposition 2.1. *Let $A \in \exp(X)$ and $f \colon A \to \mathbb{R}$ a continuous function. Then $f \in C_u^*$ if and only if there exists $\varphi_f \in \mathcal{J}$ which is the least modulus function for f.*

Proof. Sufficiency is trivial. To prove necessity suppose $f \in C_u^*$. If f is a constant function let $\varphi_f \equiv 0$. If f is not constant let

$$D_f = \bigcup_{x,y \in \mathrm{dom} f} [d(x,y), \infty) \times [0, |f(x) - f(y)|]$$

Then D_f is a subset of $[0,\infty) \times [0,\infty)$ of bounded height. We obtain $co(D_f) \cap \{0\} \times [0,\infty) = \{(0,0)\}$ because f is uniformly continuous. Let φ_f be the function whose graph $\Gamma\varphi_f$ is the upper boundary of $co(D_f)$. □

Lemma 2.2. *Let $f \in C_u^*$ and $\varepsilon > 0$. There exists $\delta > 0$ such that for $g \in C_u^*$ with $H(f,g) < \delta$, $x \in \mathrm{dom} g$, $y \in \mathrm{dom} f$, and $d(x,y) < \delta$ we have $|g(x) - f(y)| < \varepsilon$.*

Proof. Let $0 < \delta < \varepsilon/2$ so $\varphi_f(2\delta) < \varepsilon/2$. Let $g \in C_u^*$, $x \in \mathrm{dom} g$ and $y \in \mathrm{dom} f$ with $H(f,g) < \delta$ and $d(x,y) < \delta$. Let $z \in \mathrm{dom} f$ with $\tilde{d}((x,g(x)),(z,f(z))) = d(x,z) + |g(x) - f(z)| < \delta$. Then $d(z,y) < 2\delta$ so $|g(x) - f(y)| \leq |g(x) - f(z)| + |f(z) - f(y)| < \delta + \varepsilon/2 < \varepsilon$. □

Proposition 2.3. *If $\{f_i\}_{i=1}^{\infty} \subset C_u^*$ with $\lim_{i\to\infty} f_i = f_0$ in C_u^* then φ_{f_0} is the uniform limit of $\{\varphi_{f_i}\}_{i=1}^{\infty}$.*

Proof. By Lemma 2.2 $co(D_{f_0}) = \lim_{i\to\infty} co(D_{f_i})$. Since the sets $\{co(D_{f_i})\}$ converge to $co(D_{f_0})$ their upper boundaries $\{\Gamma\varphi_{f_i}\}$ converge to the upper boundary of $co(D_{f_0})$ which is $\Gamma\varphi_{f_0}$. □

For every $A \in \exp X$ denote by $\mathcal{L}(A)$ the set of all Lipschitz real-valued functions on A. It is clear that the set $\mathcal{L}(A)$ can be considered as a subspace of $C_u^*(A)$. Then the set

$$\mathcal{L} = \bigcup \{\mathcal{L}(A) : A \in \exp(X)\}$$

of all partial Lipschitz functions can be viewed as a subspace of C_u^*. For every $f \in \mathcal{L}$ let

$$\|f\|_{\mathrm{lip}} = \sup_{x,y \in \mathrm{dom} f, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}$$

denote the Lipschitz seminorm of f i.e. the greatest lower bound over all constants λ for which the inequality $|f(x) - f(y)| \leq \lambda d(x,y)$ is true for every $x, y \in \mathrm{dom} f$.

3. Extension of functions

Theorem 3.1. *There exists an operator $v \colon C_u^* \to C_u^*(X)$ which satisfies the following conditions for every $f \in C_u^*$.*

1) *$v(f)$ is an extension of f over X;*
2) *v is regular i.e. $\|v(f)\| = \|f\|$;*
3) *v is positive homogeneous i.e. $v(cf) = cv(f)$ for every $c > 0$;*
4) *v is a continuous map;*

Proof. For every $f \in C_u^*$ let φ_f be the least concave modulus function for f. Define an operator $v\colon C_u^* \to \mathbb{R}^X$ by letting

$$v(f)(x) = \min\left\{\inf_{y\in \mathrm{dom} f}(f(y) + \varphi_f(d(x,y))),\ \|f\|\right\}$$

for every $x \in X$ (here $\mathbb{R}^X$ denotes the set of all real functions on X).

It is clear that $v(f)$ is a bounded function on X for every $f \in C_u^*$. Let us prove that $v(f)$ is a uniformly continuous function on X. Let $w(f)(x) = \inf_{y\in \mathrm{dom} f}(f(y)+\varphi_f(d(x,y)))$ for every $f \in C_u^*$ and $x \in X$. To show that $w(f)$ is uniformly continuous on X let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon) > 0$ be so $\varphi_f(d(x,y)) < \varepsilon$ whenever $d(x,y) < \delta$. Take any points $x, y \in X$ with $d(x,y) < \delta$ and suppose that $w(f)(x) \geq w(f)(y)$. There exists $b \in \mathrm{dom} f$ such that $w(f)(y) + \varepsilon > f(b) + \varphi_f(d(y,b))$. We obtain

$$w(f)(y) + \varepsilon > f(b) + \varphi_f(d(y,b)) \geq f(b) + \varphi_f(d(x,b)) - \varphi_f(d(x,y)) \geq$$
$$w(f)(x) - \varphi_f(d(x,y)) > w(f)(x) - \varepsilon \geq w(f)(y) - \varepsilon.$$

Therefore, $w(f)(y)+2\varepsilon > w(f)(x) \geq w(f)(y)$. Similarly one can obtain the needed inequality for the case when $w(f)(x) < w(f)(y)$. Hence, the function $w(f)$ is uniformly continuous on X and so is $v(f) = \min\{w(f), \|f\|\}$.

Let us show that $v(f)|_{\mathrm{dom} f} = f$. If $x, y \in \mathrm{dom} f$ then $v(f)(x) \leq f(y) + \varphi_f(d(x,y))$, in particular, $v(f)(x) \leq f(x) \leq \|f\|$. Now suppose there exists $y \in \mathrm{dom} f$, $y \neq x$ such that $f(y) + \varphi_f(d(x,y)) < f(x)$. Then $f(x) - f(y) > \varphi_f(d(x,y))$ which contradicts the definition of φ_f. Therefore, $v(f)(x) = f(x)$ for $x \in \mathrm{dom} f$ and, hence, v is an extension operator.

It is easily seen that v is regular since $\|v(f)\| \leq \|f\|$ by the definition of v and $\|v(f)\| \geq \|f\|$ because $v(f)|_{\mathrm{dom} f} = f$. Thus $\|f\| = \|v(f)\|$.

To show that v is positive homogeneous note that for $c > 0$ and $f \in C_u^*$ we have $\varphi_{cf} = c\varphi_f$ and $\|cf\| = c\|f\|$. This implies $v(cf) = cv(f)$.

We are going to prove that v is a continuous operator. Suppose that $\{f_n\}$ is a sequence of functions from C_u^* which converges to $f \in C_u^*$ for some f. Let $\mathrm{dom} f_n = B_n$ and $\mathrm{dom} f = B$. Then φ_{f_n} converges to φ_f uniformly on $[0,\infty)$ by Proposition 2.3. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\varphi_f(\delta) < \varepsilon/4$. For all sufficiently large n the following conditions are satisfied:

(i) The Hausdorff distance between B and B_n is less than δ;
(ii) $|f(x) - f_n(y)| < \varepsilon/4$ for $x \in B$, $y \in B_n$ with $d(x,y) < \delta$;
(iii) $|\varphi_{f_n}(t) - \varphi_f(t)| < \varepsilon/4$ for every $t \in [0,\infty)$;
(iv) $\varphi_{f_n}(\delta) < \varepsilon/4$;
(v) $\big|\|f\| - \|f_n\|\big| < \varepsilon$.

Take any point $x \in X$ and choose arbitrary $n \in \mathbb{N}$ which satisfies (i)-(v). We are going to prove that $|v(f)(x) - v(f_n)(x)| < \varepsilon$.

If $v(f)(x) = \|f\|$ then $v(f_n)(x) \leq \|f_n\| < \|f\| + \varepsilon = v(f)(x) + \varepsilon$.

Now suppose that $v(f)(x) < \|f\|$. Then there is $y \in B$ such that $v(f)(x)+\varepsilon/4 > f(y) + \varphi_f(d(x,y))$. Let $y' \in B_n$ such that $d(y,y') < \delta$. We obtain

$$v(f_n)(x) \leq f_n(y') + \varphi_{f_n}(d(x,y')) < f(y) + \frac{\varepsilon}{4} + \varphi_{f_n}(d(x,y)) + \varphi_{f_n}(d(y,y')) <$$
$$f(y) + \frac{\varepsilon}{4} + \varphi_f(d(x,y)) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < v(f)(x) + \varepsilon.$$

Similarly one can prove that $v(f)(x) < v(f_n)(x) + \varepsilon$. This shows that v is a continuous operator. □

Corollary 3.2. *The restriction $v|_{\mathcal{L}}\colon \mathcal{L} \to C_u^*(X)$ preserves Lipschitz functions and Lipschitz constants.*

Proof. Take any function f from $\mathcal{L}$. By the definition of v we have $v(f)(x) = \min\{w(f)(x), \|f\|\}$ where

$$w(f)(x) = \inf_{z \in \mathrm{dom} f} (f(z) + \varphi_f(d(x,z))).$$

Note that for every Lipschitz function g and a constant $c \in \mathbb{R}$ the function $h(x) = \min\{g(x), c\}$ is Lipschitz with $\|h\|_{\mathrm{lip}} \leq \|g\|_{\mathrm{lip}}$. Indeed, the needed inequality is clear when $h(x) = g(x)$ and $h(y) = g(y)$ or $g(x) = c$ and $g(y) = c$ for $x, y \in X$. If $h(x) = g(x)$ and $h(y) = c$ then $g(x) \leq c \leq g(y)$. We obtain

$$|h(x) - h(y)| = |g(x) - c| = c - g(x) \leq g(y) - g(x) \leq \|g\|_{\mathrm{lip}} d(x,y).$$

So it suffices to show that $w(f)$ is a Lipschitz function on X. Let $x, y \in X$ and $\varepsilon > 0$. Take $a, b \in \mathrm{dom} f$ such that

$$w(f)(x) + \frac{\varepsilon}{3} \geq f(a) + \varphi_f(d(x,a))$$

and

$$w(f)(y) + \frac{\varepsilon}{3} \geq f(b) + \varphi_f(d(y,b)).$$

Suppose that $f(a) + \varphi_f(d(x,a)) \geq f(b) + \varphi_f(d(y,b))$. Then

$$|w(f)(x) - w(f)(y)| \leq |f(a) + \varphi_f(d(x,a)) - f(b) - \varphi_f(d(y,b)| + \frac{2\varepsilon}{3} \leq$$

$$w(f)(x) + \frac{\varepsilon}{3} - f(b) - \varphi_f(d(y,b)) + \frac{2\varepsilon}{3} \leq$$

$$f(b) + \varphi_f(d(x,b)) - f(b) - \varphi_f(d(y,b)) + \varepsilon \leq \varphi_f(d(x,y)) + \varepsilon \leq \|f\|_{\mathrm{lip}} d(x,y) + \varepsilon.$$

Since ε was arbitrary we obtain the needed inequality for $w(f)$ and therefore for $v(f)$. It is obvious that $\|v(f)\|_{\mathrm{lip}} \geq \|f\|_{\mathrm{lip}}$ because $v(f)$ is an extension of f. Therefore, we obtain $\|v(f)\|_{\mathrm{lip}} = \|f\|_{\mathrm{lip}}$. □

4. Extension of metrics

For every $A \in \exp(X)$ with $|A| \geq 2$ denote by $\mathcal{M}(A)$ the set of all uniformly continuous bounded metrics on A. Similarly as for functions we assume that each partial metric is identified with its graph which is a closed and bounded subset of the space $X \times X \times \mathbb{R}$. Hence the set of all uniformly continuous partial metrics

$$\mathcal{M} = \bigcup \{\mathcal{M}(A) : A \in \exp(X),\ |A| \geq 2\}$$

is considered as a subspace of the metric space $(\exp(X \times X \times \mathbb{R}), H)$ (here H denotes the Hausdorff metric, generated by the l_1 metric on $X \times X \times \mathbb{R}$). We will write $\mathrm{dom}\rho = A$ if $\rho \in \mathcal{M}(A)$. For every $\rho \in \mathcal{M}$ let $\|\rho\| = \sup\{\rho(x,y) : x, y \in \mathrm{dom}\rho\}$. Clearly, the counterparts of Lemma 2.2 and Proposition 2.3 are also true for the case of metrics.

For every $A \in \exp X$ with $|A| \geq 2$ denote by $\mathcal{LM}(A)$ the set of all Lipschitz metrics on A. Recall that a metric $\rho \in \mathcal{M}(A)$ is Lipschitz if there is a constant $\lambda > 0$ such that $\rho(x,y) \leq \lambda d(x,y)$ for all $x, y \in A$. Then the set $\mathcal{LM}(A)$ can be viewed as a subspace of $\mathcal{M}(A)$ and the set

$$\mathcal{LM} = \bigcup \{\mathcal{LM}(A) : A \in \exp(X), |A| \geq 2\}$$

is a subspace of $\mathcal{M}$. For every $\rho \in \mathcal{LM}$ let

$$\|\rho\|_{\mathrm{lip}} = \sup_{x,y\in \mathrm{dom}\rho, x\neq y} \frac{\rho(x,y)}{d(x,y)}$$

Theorem 4.1. *Let (X,d) be a bounded metric space. There exists an operator $u\colon \mathcal{M} \to \mathcal{M}(X)$ which has the following properties for every $\rho \in \mathcal{M}$.*

1) *$u(\rho)$ is an extension of ρ over X;*
2) *u is regular that is $\|u(\rho)\| = \|\rho\|$;*
3) *u is positive homogeneous that is $u(c\rho) = cu(\rho)$ for every $c > 0$;*
4) *u is a continuous map;*
5) *the restriction $u|_{\mathcal{LM}}\colon \mathcal{LM} \to \mathcal{M}(X)$ preserves Lipschitz metrics and Lipschitz norms.*

Proof. For $\rho \in \mathcal{M}$ let φ_ρ be the smallest concave modulus function of ρ i.e. $\varphi_\rho\colon [0,\infty) \to [0,\infty)$ is the least concave function such that $\varphi_\rho(t) \geq \sup\{\rho(x,y) : x,y \in \mathrm{dom}\rho, d(x,y) \leq t\}$. Define a metric $\sigma_\rho\colon X\times X \to \mathbb{R}$ by letting $\sigma_\rho(x,y) = \varphi_\rho(d(x,y))$ for all $x,y \in X$. Clearly, $\sigma_\rho(x,x) = 0$, σ_ρ is symmetric and satisfies the triangle inequality because d is a metric and φ_ρ is concave and non-decreasing.

For $x,y \in X$ let

$$u(\rho)(x,y) = \min\left\{\inf_{a,b\in\mathrm{dom}\rho}\Big(\sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(b,y)\Big), \sigma_\rho(x,y)\right\}.$$

Let us show that $u(\rho) \in \mathcal{M}(X)$ and $u(\rho)|_{\mathrm{dom}\rho\times\mathrm{dom}\rho} = \rho$. It is easy to see that $u(\rho)$ is symmetric and $u(\rho)(x,x) = 0$ since $\sigma_\rho(x,x) = 0$ for every $x \in X$. Note also that $u(\rho)(x,y) > 0$ for $x \neq y$. To show that $u(\rho)$ satisfies the triangle inequality let $x,y,z \in X$. Consider several cases.

(a) Suppose $u(\rho)(x,z) = \sigma_\rho(x,z)$ and $u(\rho)(z,y) = \sigma_\rho(z,y)$. We obtain

$$u(\rho)(x,y) \leq \sigma_\rho(x,y) \leq \sigma_\rho(x,z) + \sigma_\rho(z,y) = u(\rho)(x,z) + u(\rho)(z,y).$$

(b) Suppose $u(\rho)(x,z) < \sigma_\rho(x,z)$ and $u(\rho)(z,y) = \sigma_\rho(z,y)$. Let $\varepsilon > 0$ and $a,b \in \mathrm{dom}\rho$ be such that $\sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(z,b) < u(\rho)(x,z) + \varepsilon$. Then

$$u(\rho)(x,z) + \varepsilon + u(\rho)(z,y) > \sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(z,b) + \sigma_\rho(z,y) \geq$$

$$\sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(y,b) \geq u(\rho)(x,y).$$

Since $\varepsilon > 0$ was arbitrary we obtain $u(\rho)(x,y) \leq u(\rho)(x,z) + u(\rho)(z,y)$.

(c) Suppose $u(\rho)(x,z) < \sigma_\rho(x,z)$ and $u(\rho)(z,y) < \sigma_\rho(z,y)$. Let $\varepsilon > 0$ and $a,b,a',b' \in \mathrm{dom}\rho$ be such that $\sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(b,z) < u(\rho)(x,z) + \varepsilon/2$ and $\sigma_\rho(z,a') + \rho(a',b') + \sigma_\rho(b',y) < u(\rho)(z,y) + \varepsilon/2$. We obtain

$$\begin{gathered} u(\rho)(x,y) \leq \sigma_\rho(x,a) + \rho(a,b') + \sigma_\rho(b',y) \leq \\ \sigma_\rho(x,a) + \rho(a,b) + \rho(b,a') + \rho(a',b') + \sigma_\rho(b',y) \leq \\ \sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(b,a') + \rho(a',b') + \sigma_\rho(b',y) \leq \\ \sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(b,z) + \sigma_\rho(z,a') + \rho(a',b') + \sigma_\rho(b',y) \leq \\ u(\rho)(x,z) + \frac{\varepsilon}{2} + u(\rho)(z,y) + \frac{\varepsilon}{2}. \end{gathered}$$

Since $\varepsilon > 0$ was arbitrary we obtain $u(\rho)(x,y) \leq u(\rho)(x,z) + u(\rho)(z,y)$. Therefore, $u(\rho)$ is a metric on X.

It is easy to see that $u(\rho)$ is uniformly continuous for every $\rho \in \mathcal{M}$. Take any $\varepsilon > 0$. There exists $\delta = \delta(\varepsilon) > 0$ such that $\varphi_\rho(d(x,y)) < \varepsilon$ whenever $d(x,y) < \varepsilon$.

Then for every pair of points $x, y \in X$ with $d(x,y) < \delta$ we have $u(\rho)(x,y) \le \sigma_\rho(x,y) = \varphi_\rho(d(x,y)) < \varepsilon$. The boundedness of $u(\rho)$ is an easy consequence of the definition of u.

Let $x, y, a, b \in \mathrm{dom}\rho$. Then $\sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(b,y) \ge \rho(x,a) + \rho(a,b) + \rho(b,y) \ge \rho(x,y)$. Also $\rho(x,y) \le \sigma_\rho(x,y)$ by the definition of σ_ρ. So $u(\rho)(x,y) = \rho(x,y)$. Hence $u(\rho)$ is a metric which extends ρ over X.

It is clear that v is regular since for every $\rho \in \mathcal{M}$ $\|u(\rho)\| \le \|\sigma_\rho\| = \|\rho\|$ by the definition of u and $\|u(\rho)\| \ge \|\rho\|$ because $u(\rho)|_{\mathrm{dom}\rho \times \mathrm{dom}\rho} = \rho$. Thus $\|\rho\| = \|u(\rho)\|$.

To show that u is positive homogeneous note that for $c > 0$ and $\rho \in \mathcal{M}$ we have $\varphi_{c\rho} = c\varphi_\rho$. This implies $u(c\rho) = cu(\rho)$.

Let us prove that the operator u is continuous. Let $\{\rho_n\}$ be a sequence in $\mathcal{M}$ which converges to some $\rho \in \mathcal{M}$. Let $\mathrm{dom}\rho_n = B_n$, $\mathrm{dom}\rho = B$ and $\varepsilon > 0$. Choose $\delta > 0$ such that $\varphi_\rho(\delta) < \frac{\varepsilon}{16}$. Then for sufficiently large n the following conditions are satisfied

(1) The Hausdorff distance between B and B_n is less than $\delta/4$;
(2) $|\rho(x,y) - \rho_n(x',y')| < \varepsilon/16$ for $x, y \in B$, $x', y' \in B_n$ with $d(x,x') < \delta$ and $d(y,y') < \delta$;
(3) $|\sigma_\rho(x,y) - \sigma_{\rho_n}(x,y)| < \varepsilon/16$ for all $x, y \in X$ since φ_{ρ_n} converges to φ_ρ uniformly;
(4) $\varphi_{\rho_n}(\delta) < \varepsilon/16$.

Take any pair of points $x, y \in X$ and choose arbitrary $n \in \mathbb{N}$ which satisfies (1)-(4). We are going to prove that $|u(\rho)(x,y) - u(\rho_n)(x,y)| < \varepsilon$. Consider several cases.

Case 1. Suppose $x, y \in B$. Then $u(\rho)(x,y) = \rho(x,y)$. Let $x_n, y_n \in B_n$ be such that $d(x,x_n) < \delta$ and $d(y,y_n) < \delta$. We obtain

$$u(\rho_n)(x,y) \le \sigma_{\rho_n}(x,x_n) + \rho_n(x_n,y_n) + \sigma_{\rho_n}(y_n,y) <$$

$$\varphi_{\rho_n}(\delta) + \rho(x,y) + \frac{\varepsilon}{16} + \varphi_{\rho_n}(\delta) < \rho(x,y) + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} < u(\rho)(x,y) + \varepsilon.$$

Now

$$u(\rho)(x,y) = \rho(x,y) < \rho_n(x_n,y_n) + \frac{\varepsilon}{16} \le$$

$$u(\rho_n)(x,y) + \sigma_{\rho_n}(x,x_n) + \sigma_{\rho_n}(y,y_n) + \frac{\varepsilon}{16} \le$$

$$u(\rho_n)(x,y) + \varphi_{\rho_n}(\delta) + \varphi_{\rho_n}(\delta) + \frac{\varepsilon}{16} < u(\rho_n)(x,y) + \varepsilon.$$

Case 2. Suppose $x \in B$ and $y \notin B$. We prove first that $u(\rho_n)(x,y) < u(\rho)(x,y) + \varepsilon$. Take $b \in B$ such that $\sigma_\rho(y,b) + \rho(b,x) < u(\rho)(x,y) + \varepsilon/16$. Choose points $b_n \in B_n$ and $x_n \in B_n$ such that $d(x,x_n) < \delta$, $d(b,b_n) < \delta$. We obtain

$$u(\rho_n)(x,y) \le \sigma_{\rho_n}(x,x_n) + \rho_n(x_n,b_n) + \sigma_{\rho_n}(b_n,y) \le$$

$$\sigma_{\rho_n}(x,x_n) + \rho_n(x_n,b_n) + \sigma_{\rho_n}(b,b_n) + \sigma_{\rho_n}(b,y) <$$

$$\varphi_{\rho_n}(\delta) + \rho(x,b) + \frac{\varepsilon}{16} + \varphi_{\rho_n}(\delta) + \sigma_\rho(b,y) + \frac{\varepsilon}{16} <$$

$$\frac{\varepsilon}{16} + \rho(x,b) + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \sigma_\rho(b,y) + \frac{\varepsilon}{16} = \rho(x,b) + \sigma_\rho(b,y) + \frac{\varepsilon}{4} < u(\rho)(x,y) + \varepsilon.$$

Now we prove that $u(\rho)(x,y) < u(\rho_n)(x,y)+\varepsilon$. We may suppose that $x,y \notin B_n$ or we are back in the first part of Case 2 or in Case 1 with the roles of ρ_n and ρ interchanged. Suppose first that $u(\rho_n)(x,y) = \sigma_{\rho_n}(x,y)$. Then

$$u(\rho)(x,y) \le \sigma_\rho(x,y) < \sigma_{\rho_n}(x,y) + \frac{\varepsilon}{16} < u(\rho_n)(x,y)+\varepsilon.$$

Now let $u(\rho_n)(x,y) < \sigma_{\rho_n}(x,y)$. There are points $a_n, b_n \in B_n$ such that

$$u(\rho_n)(x,y) + \frac{\varepsilon}{8} > \sigma_{\rho_n}(x,a_n) + \rho_n(a_n,b_n) + \sigma_{\rho_n}(b_n,y).$$

Choose $b \in B$ and $x_n \in B_n$ such that $d(x,x_n) < \delta$ and $d(b,b_n) < \delta$. Note that

$$\rho_n(a_n,x_n) \le \sigma_{\rho_n}(a_n,x_n) \le \sigma_{\rho_n}(x,x_n) + \sigma_{\rho_n}(x,a_n) < \sigma_{\rho_n}(x,a_n) + \frac{\varepsilon}{16}.$$

We obtain

$$u(\rho)(x,y) \le \sigma_\rho(b,y) + \rho(b,x) \le \sigma_\rho(y,b_n) + \sigma_\rho(b,b_n) + \rho(b,x) <$$

$$\sigma_{\rho_n}(y,b_n) + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \rho_n(x_n,b_n) + \frac{\varepsilon}{16} \le$$

$$\sigma_{\rho_n}(y,b_n) + \rho_n(x_n,a_n) + \rho_n(a_n,b_n) + \frac{3\varepsilon}{16} <$$

$$\sigma_{\rho_n}(y,b_n) + \sigma_{\rho_n}(x,a_n) + \frac{\varepsilon}{16} + \rho_n(a_n,b_n) + \frac{3\varepsilon}{16} < u(\rho_n)(x,y)+\varepsilon.$$

Case 3. Suppose $x,y \notin B \cup B_n$. First suppose $u(\rho)(x,y) < \sigma_\rho(x,y)$. Choose $a,b \in B$ such that

$$u(\rho)(x,y) + \frac{\varepsilon}{2} > \sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(y,b).$$

Take $a_n, b_n \in B_n$ such that $d(a,a_n) < \delta$ and $d(b,b_n) < \delta$. We obtain

$$u(\rho_n)(x,y) \le \sigma_{\rho_n}(x,a_n) + \rho_n(a_n,b_n) + \sigma_{\rho_n}(y,b_n) \le$$

$$\sigma_{\rho_n}(x,a) + \sigma_{\rho_n}(a,a_n) + \rho_n(a_n,b_n) + \sigma_{\rho_n}(y,b) + \sigma_{\rho_n}(b,b_n) <$$

$$\sigma_{\rho_n}(x,a) + \frac{\varepsilon}{16} + \rho_n(a_n,b_n) + \frac{\varepsilon}{16} + \sigma_{\rho_n}(y,b) <$$

$$\sigma_\rho(x,a) + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \rho(a,b) + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \sigma_\rho(y,b) + \frac{\varepsilon}{16} < u(\rho)(x,y)+\varepsilon.$$

Now let $u(\rho)(x,y) = \sigma_\rho(x,y)$. Then

$$u(\rho_n)(x,y) \le \sigma_{\rho_n}(x,y) < \sigma_\rho(x,y) + \frac{\varepsilon}{16} < u(\rho)(x,y)+\varepsilon.$$

Similarly interchanging the roles of ρ and ρ_n we can prove that $u(\rho)(x,y) < u(\rho_n)(x,y)+\varepsilon$.

To prove that the operator u preserves Lipschitz metrics let $\rho \in \mathcal{LM}$. Then

$$u(\rho)(x,y) \le \sigma_\rho(x,y) = \varphi_\rho(d(x,y)) \le \|\rho\|_{\mathrm{lip}} d(x,y)$$

by Lipschitzness of ρ and the definition of φ_ρ. □

5. Extension of ultrametrics

In this section we are going to use a modification of the extension operator for uniformly continuous metrics to obtain its counterpart for uniformly continuous partial ultrametrics defined on closed subsets of a bounded ultrametric space. It is proved in [11] that there exists a continuous positive homogeneous extension operator for the family of all uniformly continuous ultrametrics defined on closed subsets of a bounded complete ultrametric space. This operator has an additional property of preserving maxima of ultrametrics with a common domain. The operator constructed in this section in general does not preserve maxima but preserves Lipschitz ultrametrics. Moreover, we drop the completeness assumption for the space (X, d).

Recall that a metric r on a set Y is called an ultrametric if it satisfies the strong triangle inequality i.e. $r(x, y) \le \max\{r(x, z), r(y, z)\}$ for all $x, y, z \in Y$. Let (X, d) be a bounded ultrametric space. For every $A \in \exp X$ with $|A| \ge 2$ denote by $\mathcal{UM}(A)$ the set of all uniformly continuous bounded ultrametrics defined on A. Similarly as before, $\mathcal{UM} = \bigcup\{\mathcal{UM}(A) : A \in \exp(X), |A| \ge 2\}$ stands for the set of all partial ultrametrics. It is clear that $\mathcal{UM}$ can be viewed as a subspace of the space $\mathcal{M}$ of all uniformly continuous bounded metrics defined on closed subsets of the ultrametric space (X, d).

Theorem 5.1. *Let (X, d) be a bounded ultrametric space. There exists an operator $\alpha\colon \mathcal{UM} \to \mathcal{UM}(X)$ which has the following properties for every $\rho \in \mathcal{UM}$.*

1) *$\alpha(\rho)$ is an extension of ρ over X;*
2) *α is regular that is $\|\alpha(\rho)\| = \|\rho\|$;*
3) *α is positive homogeneous that is $\alpha(c\rho) = c\alpha(\rho)$ for every $c > 0$;*
4) *α is a continuous map;*
5) *α preserves Lipschitz ultrametrics and Lipschitz norms.*

Proof. Recall that for $\rho \in \mathcal{UM}$ φ_ρ denotes the smallest concave modulus function for ρ and $\sigma_\rho\colon X \times X \to \mathbb{R}$ is a metric on X defined by $\sigma_\rho(x, y) = \varphi_\rho(d(x, y))$ for all $x, y \in X$. It is easy to see that σ_ρ is in fact an ultrametric. Indeed

$$\sigma_\rho(x, y) = \varphi_\rho(d(x, y)) \le \varphi_\rho(\max\{d(x, z), d(z, y)\}) =$$
$$\max\{\varphi_\rho(d(x, z)), \varphi_\rho(d(y, z))\} = \max\{\sigma_\rho(x, z), \sigma_\rho(y, z)\}.$$

For $x, y \in X$ let

$$\alpha(\rho)(x, y) = \min\left\{\inf_{a,b\in\mathrm{dom}\rho} \max\{\sigma_\rho(x, a), \rho(a, b), \sigma_\rho(b, y)\}, \sigma_\rho(x, y)\right\}.$$

One can show that $\alpha(\rho) \in \mathcal{UM}(X)$ in a similar way as for the case of metrics. It is easy to see that $\alpha(\rho)$ is symmetric and $\alpha(\rho)(x, x) = 0$ since $\sigma_\rho(x, x) = 0$ for every $x \in X$. Note also that $\alpha(\rho)(x, y) > 0$ for $x \ne y$. To show that $\alpha(\rho)$ satisfies the strong triangle inequality let $x, y, z \in X$. Consider several cases.

(a) Suppose $\alpha(\rho)(x, z) = \sigma_\rho(x, z)$ and $\alpha(\rho)(z, y) = \sigma_\rho(z, y)$. We obtain

$$\alpha(\rho)(x, y) \le \sigma_\rho(x, y) \le \max\{\sigma_\rho(x, z), \sigma_\rho(z, y)\} = \max\{\alpha(\rho)(x, z), \alpha(\rho)(z, y)\}.$$

(b) Suppose $\alpha(\rho)(x, z) < \sigma_\rho(x, z)$ and $\alpha(\rho)(z, y) = \sigma_\rho(z, y)$. Let $\varepsilon > 0$ and $a, b \in \mathrm{dom}\rho$ be such that $\max\{\sigma_\rho(x, a), \rho(a, b), \sigma_\rho(z, b)\} < \alpha(\rho)(x, z) + \varepsilon$. Then

$$\max\{\alpha(\rho)(x, z) + \varepsilon, \alpha(\rho)(z, y)\} > \max\{\sigma_\rho(x, a), \rho(a, b), \sigma_\rho(z, b), \sigma_\rho(z, y)\} \ge$$
$$\max\{\sigma_\rho(x, a), \rho(a, b), \sigma_\rho(y, b)\} \ge \alpha(\rho)(x, y).$$

Since $\varepsilon > 0$ was arbitrary we obtain $\alpha(\rho)(x, y) \le \max\{\alpha(\rho)(x, z), \alpha(\rho)(z, y)\}$.

(c) Suppose $\alpha(\rho)(x,z) < \sigma_\rho(x,z)$ and $\alpha(\rho)(z,y) < \sigma_\rho(z,y)$. Let $\varepsilon > 0$ and $a,b,a',b' \in \mathrm{dom}\rho$ be such that $\max\{\sigma_\rho(x,a),\rho(a,b),\sigma_\rho(b,z)\} < \alpha(\rho)(x,z)+\varepsilon$ and $\max\{\sigma_\rho(z,a'),\rho(a',b'),\sigma_\rho(b',y)\} < \alpha(\rho)(z,y)+\varepsilon$. We obtain

$$\begin{gathered}\alpha(\rho)(x,y) \le \max\{\sigma_\rho(x,a),\rho(a,b'),\sigma_\rho(b',y)\} \le \\ \max\{\sigma_\rho(x,a),\rho(a,b),\rho(b,a'),\rho(a',b'),\sigma_\rho(b',y)\} \le \\ \max\{\sigma_\rho(x,a),\rho(a,b),\sigma_\rho(b,a'),\rho(a',b'),\sigma_\rho(b',y)\} \le \\ \max\{\sigma_\rho(x,a),\rho(a,b),\sigma_\rho(b,z),\sigma_\rho(z,a'),\rho(a',b'),\sigma_\rho(b',y)\} \le \\ \max\{\alpha(\rho)(x,z)+\varepsilon,\alpha(\rho)(z,y)+\varepsilon\}.\end{gathered}$$

Since $\varepsilon > 0$ was arbitrary we obtain $\alpha(\rho)(x,y) \le \max\{\alpha(\rho)(x,z),\alpha(\rho)(z,y)\}$. Therefore, $\alpha(\rho)$ is an ultrametric on X.

The remaining properties of the operator α can be proved similarly to those of the extension operator from the previous theorem.

□

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