A continuous extension operator for convex metrics

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Abstract

We consider the problem of simultaneous extension of continuous convex metrics defined on subcontinua of a Peano continuum. We prove that there is an extension operator for convex metrics which is continuous with respect to the uniform topology.

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1 Introduction

The following theorem is the main result of this paper. For X a Peano continuum $\mathcal{CM}(X)$ is the set of convex metrics on X. Then

$$\mathcal{CM} = \bigcup\{\mathcal{CM}(A) : A \text{ is a Peano subcontinuum of } X\}$$

is the set of all partial convex metrics. The distance between two members of $\mathcal{CM}$ is the Hausdorff distance between their graphs.

Theorem. *There exists a continuous extension operator* $u\colon \mathcal{CM} \to \mathcal{CM}(X)$.

Bing [6] had proved that there is an extension operator $u\colon \mathcal{CM} \to \mathcal{CM}(X)$ but his operator was not continuous. We follow Bing's argument with the one essential addition in that we choose in a canonical way a modulus function for each member of $\mathcal{CM}$. The bulk of this paper is devoted to the proof of continuity of our operator.

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The problem of extension of continuous functions and (pseudo)metrics defined on a closed subset of a topological (metrizable) space goes back to Tietze, Urysohn and Hausdorff. The theory of continuous extensions for pseudometrics has developed in parallel with the theory of continuous extensions for real-valued functions (see [4]). Dugundji (1951) had proved that there exists a continuous linear operator extending real-valued continuous functions defined on a closed subset of a metric space. C. Bessaga raised the question of existence of such operators for the cone of (pseudo)metrics and gave the solution for some special cases [5]. The complete solution of this problem was obtained by T. Banakh [1]. E. Stepanova [12] was first to consider the problem of extension of continuous functions with variable domains. The set of continuous pseudometrics on a compact metrizable space X forms a positive cone in the normed space $C(X\times X)$ of continuous real-valued functions on $X \times X$. Recently E. D. Tymchatyn and M. Zarichnyi [13] proved that there exists a continuous linear operator which extends simultaneously continuous pseudometrics defined on closed subsets of a metrizable compact topological space.

2 Preliminaries

Let (X, r) be a metric space and $\exp(X)$ be the family of bounded, closed and non-empty subsets of X. For $A \in \exp(X)$ let $\mathcal{PM}(A)$ denote the family of bounded pseudometrics on A which are continuous with respect to r and let

$$\mathcal{PM} = \bigcup\{\mathcal{PM}(A) : A \in \exp(X)\}.$$

If $\rho \in \mathcal{PM}(A)$ we denote A by $\mathrm{dom}\rho$. We identify each $\rho \in \mathcal{PM}$ with its graph

$$\Gamma_\rho = \{(x, y, \rho(x, y)) : x, y \in \mathrm{dom}\rho\}.$$

Then $\Gamma_\rho \in \exp(X \times X \times \mathbb{R})$. We let $\exp(X \times X \times \mathbb{R})$ have the Hausdorff metric derived from the metric r' on $X \times X \times \mathbb{R}$, where

$$r'[(x, y, z), (x', y', z')] = r(x, x') + r(y, y') + |z - z'|$$

for $x, y, x', y' \in X$ and $z, z' \in \mathbb{R}$. Then $\mathcal{PM}$ is a subspace of the space $\exp(X \times X \times \mathbb{R})$.

A *continuous extension operator* $u\colon \mathcal{PM} \to \mathcal{PM}(X)$ is a continuous map such that for $\rho \in \mathcal{PM}$ $u(\rho)$ is a continuous pseudometric on X with

$u(\rho)|_{\mathrm{dom}\rho\times\mathrm{dom}\rho} = \rho$. We say u is linear if $u(\alpha\rho + \beta\tau) = \alpha u(\rho) + \beta u(\tau)$ for $\alpha, \beta \geq 0$ and $\rho, \tau \in \mathcal{PM}(A)$ for $A \in \exp(X)$. For $\rho \in \mathcal{PM}$ let $\|\rho\| = \sup\{\rho(x, y) : x, y \in \mathrm{dom}\rho\}$. Recently the second named author and Zarichnyi [13] considered the problem of simultaneous extension of pseudometrics for a compact space.

Theorem 1([13]). *Let X be a compact metrizable space. There exists a continuous linear extension operator $u\colon \mathcal{PM} \to \mathcal{PM}(X)$ such that $\|u(\rho)\| = \|\rho\|$ for each $\rho \in \mathcal{PM}$.*

It is natural to ask for versions of Theorem 1 for important subclasses of pseudometrics. There has been some success in getting continuous extension operators for the subclass of partial ultrametrics on a metric compactum. A metric compactum (X, r) admits an ultrametric if and only if it is zero-dimensional. For $A \in \exp(X)$ let $\mathcal{UM}(A)$ denote the family of ultrametrics on A which are continuous with respect to r and let

$$\mathcal{UM} = \bigcup\{\mathcal{UM}(A) : A \in \exp(X),\ |A| \geq 2\} \subset \mathcal{PM}.$$

Since the sum of two ultrametrics need not be an ultrametric, there is no sense to consider linear operators extending ultrametrics. However, the pointwise maximum of two ultrametrics with a common domain is an ultrametric.

Theorem 2([14], [11]). *Let (X, r) be a zero-dimensional metric compactum. There exists a continuous extension operator*

$$u\colon \mathcal{UM} \to \mathcal{UM}(X)$$

such that $\|u(\rho)\| = \|\rho\|$ for each $\rho \in \mathcal{UM}$ and

$$u(\max\{\rho, \rho'\}) = \max\{u(\rho), u(\rho')\} \text{ and } u(c\rho) = cu(\rho)$$

for $\rho, \rho' \in \mathcal{UM}$ with $\mathrm{dom}\rho = \mathrm{dom}\rho'$ and $c > 0$.

3 Main result.

A metric r on a Peano continuum X is said to be *convex* if for each $x, y \in X$ there is an arc $[xy]$ with endpoints x and y such that $[xy]$ is isometric to the closed interval $[0, r(x, y)]$ in the real line $\mathbb{R}$. We call such an arc an r-segment. It is known Bing [7] and Moise [9] (see also [10]) that a metric continuum is locally connected if and only if it has a convex metric.

If X is a Peano continuum and A is a locally connected subcontinuum of X let $\mathcal{CM}(A)$ denote the set of continuous convex metrics on A. Let

$$\mathcal{CM} = \bigcup\{\mathcal{CM}(A) : A \text{ is a Peano subcontinuum of } X\} \subset \mathcal{PM}.$$

It follows from Bing [7] that there is an extension operator from $\mathcal{CM}$ to $\mathcal{CM}(X)$. We prove that this extension operator can be taken to be continuous. The sum of convex metrics need not be convex so there is no hope of finding such an operator which is also linear. The following theorem is the main result of this note.

Theorem. *Let X be a Peano continuum. There is a continuous extension operator $u\colon \mathcal{CM} \to \mathcal{CM}(X)$.*

Proof. The initial part of the proof follows closely the argument of Bing [7, Theorem 1]. Let r be a convex metric for X with $\|r\| = 1$. For $\rho \in \mathcal{CM}$ let $\varphi(\rho)$ be the smallest concave modulus function for ρ i.e. $\varphi(\rho)\colon [0,1] \to [0,+\infty)$ is the least concave function such that

$$\varphi(\rho)(t) \geq \max\{\rho(x,y) : x, y \in \mathrm{dom}\rho,\ r(x,y) \leq t\}.$$

Then $\varphi(\rho)(0) = \lim_{t\to 0+} \varphi(\rho)(t)$. Also, the left derivative $\varphi(\rho)'_-(t)$ (respectively the right derivative $\varphi(\rho)'_+(t)$) is defined for each $t \in (0,1]$ (respectively $t \in (0,1)$), $\varphi(\rho)'_-(t) = \varphi(\rho)'_+(t) = \varphi(\rho)'(t)$ for all but countably many $t \in (0,1)$ and $\varphi(\rho)'_-(t)$ is non-increasing.

To visualize $\varphi(\rho)$ let

$$D_\rho = \bigcup_{x,y\in\mathrm{dom}\rho} [r(x,y),1] \times [0,\rho(x,y)].$$

Then D_ρ is a closed subset of the plane of height $\|\rho\|$. If $\mathrm{co}(D_\rho)$ is the convex hull of D_ρ then the graph of $\varphi(\rho)$ is the upper boundary of $\mathrm{co}(D_\rho)$. It is easy to see [3] that the function $\varphi\colon \mathcal{CM} \to C([0,1])$ is continuous where $C([0,1])$ is equipped with the topology of uniform convergence.

Define a continuous function $\theta\colon \mathcal{CM} \to C([0,1])$ by

$$\theta(\rho)(t) = \begin{cases} \varphi(\rho)(t) + t(1 - \varphi(\rho)'_-(1)) \text{ if } \varphi(\rho)'_-(1) < 1; \\ \varphi(\rho)(t) \text{ if } \varphi(\rho)'_-(1) \geq 1 \end{cases}$$

for $\rho \in \mathcal{CM}$ and $t \in [0,1]$. Then $\theta(\rho)$ is a concave modulus function for the metric ρ with $\theta(\rho)(0) = 0$ and $\theta(\rho)'_-(t) \geq 1$ for all $t \in (0,1]$.

Since $\theta(\rho)$ is concave and $\theta(\rho)(0) = 0$ there is a non-increasing function $\nu\colon [0,1] \to [0,\infty)$ such that $\theta(\rho)(t_0) = \int_0^{t_0} \nu(t)dt$ for $\rho \in \mathcal{CM}$ and $t_0 \in [0,1]$ by [8, 18.43]. By [8, 18.17] θ is absolutely continuous and $\nu(t) = \theta(\rho)'_-(t)$ almost everywhere on $[0,1]$. So

$$\int_0^{t_0} \theta(\rho)'_-(t)dt = \theta(\rho)(t_0) \text{ for } \rho \in \mathcal{CM} \text{ and } t_0 \in [0,1]. \tag{1}$$

If $\{\rho_i\}$ converges to ρ in $\mathcal{CM}$ and $t_0 \in (0,1]$ then $\lim_{i\to+\infty} \theta(\rho_i)'_-(t_0) = \theta(\rho)'_-(t_0)$ by continuity of θ, (1) and the left continuity of $\theta(\rho_i)'_-$ and $\theta(\rho)'_-$.

Note that $\theta(r)$ is the identity on $[0,1]$. If C is an r-rectifiable path in X let $L_r(C)$ denote its r-length. For $\rho \in \mathcal{CM}$ let $\mathcal{A}_\rho$ denote the collection of all r-rectifiable paths which meet $\mathrm{dom}\rho$ in at most their endpoints. For $C \in \mathcal{A}_\rho$ define the ρ-length of C by

$$L_\rho(C) = \int_C \theta(\rho)'_-(r(p(s), \mathrm{dom}\rho))ds$$

where C is parametrized by its arc length with respect to r and $p(s)$ is a point on C whose distance along C from a fixed endpoint of C is s. Note that

$$L_\rho(C) \geq L_r(C) \tag{2}$$

since $\theta(\rho)'_- \geq 1$.

For $x, y \in X$ let

$$\mathcal{A}_\rho(x,y) = \{C \in \mathcal{A}_\rho : x, y \text{ are the endpoints of } C\}.$$

If $\mathcal{A}_\rho(x,y) \neq \varnothing$ let $\sigma_\rho(x,y) = \inf\{L_\rho(C) : C \in \mathcal{A}_\rho(x,y)\}$ and let $\sigma_\rho(x,y) = \infty$ if $\mathcal{A}_\rho(x,y) = \varnothing$. Then $\sigma_\rho(x,y) \geq r(x,y)$ by (2).

Let $x, y \in \mathrm{dom}\rho$ such that $\sigma_\rho(x,y) < \infty$. We show that $\sigma_\rho(x,y) \geq \rho(x,y)$. To see this let $\varepsilon > 0$. By the definition of σ_ρ there exists $C \in \mathcal{A}_\rho(x,y)$ such that

$$\sigma_\rho(x,y) + \varepsilon > L_\rho(C) = \int_C \theta(\rho)'_-(r(p(s), \mathrm{dom}\rho))ds \geq \tag{3}$$

$$\int_C \theta(\rho)'_-(s)ds = \int_0^{L_r(C)} \theta(\rho)'_-(s)ds = \theta(\rho)(L_r(C)) \geq \theta(\rho)(r(x,y)) \geq \rho(x,y).$$

The first $\geq$ in (3) is because $\theta(\rho)'_-$ is non-increasing and $r(p(s), \mathrm{dom}\rho) \leq s$ since the endpoints of C lie in $\mathrm{dom}\rho$. The second $\geq$ in (3) is because $\theta(\rho)$ is non-decreasing. The third $\geq$ in (3) is because $\theta(\rho)$ is a modulus function for ρ. Since ε is arbitrary, we obtain the needed inequality.

If $y \in X \setminus \mathrm{dom}\rho$ and C is a shortest r-segment from y to $\mathrm{dom}\rho$ then by (1),

$$L_\rho(C) = \int_C \theta(\rho)'_-(r(p(s), \mathrm{dom}\rho))ds = \int_C \theta(\rho)'_-(s)ds = \theta(\rho)(r(y, \mathrm{dom}\rho)). \tag{4}$$

Hence, if $\{p_i\}$ is a sequence of points in $Y \setminus \mathrm{dom}\rho$ with $r(p_i, \mathrm{dom}\rho) \to 0$ as $i \to +\infty$ then $\sigma_\rho(p_i, q_i) \to 0$ as $i \to +\infty$ where $q_i \in \mathrm{dom}\rho$ are such that $r(p_i, q_i) = r(p_i, \mathrm{dom}\rho)$.

For $x, y \in X$ define

$$u(\rho)(x,y) = \begin{cases} \rho(x,y) \text{ if } x,y \in \mathrm{dom}\rho; \\ \alpha_\rho(x,y) \text{ if } |\{x,y\} \cap \mathrm{dom}\rho| = 1; \\ \min\{\sigma_\rho(x,y), \beta_\rho(x,y)\} \text{ if } x,y \in X \setminus \mathrm{dom}\rho \end{cases}$$

where

$$\alpha_\rho(b,c) = \alpha_\rho(c,b) = \inf\{\sigma_\rho(c,a) + \rho(a,b) : a \in \mathrm{dom}\rho\}$$

for $b \in \mathrm{dom}\rho$, $c \in X \setminus \mathrm{dom}\rho$ and

$$\beta_\rho(x,y) = \inf\{\sigma_\rho(x,a) + \rho(a,b) + \sigma_\rho(b,y) : a,b \in \mathrm{dom}\rho\}.$$

This definition of $u(\rho)(x,y)$ is equivalent to defining $u(\rho)(x,y)$ to be the greatest lower bound of lengths of all arcs C from x to y where length in $\mathrm{dom}\rho$ is measured by ρ and length in $X \setminus \mathrm{dom}\rho$ is measured by L_ρ. Since for $a, b \in \mathrm{dom}\rho$ $\rho(a,b) \leq \sigma_\rho(a,b)$ by (3), we need to consider only arcs which meet $\mathrm{dom}\rho$ in a connected set if at all.

By choosing r-segments outside $\mathrm{dom}\rho$ we see that $u(\rho)(x,y) \leq \|\rho\| + 2\theta(\rho)(1)$ by (4). Hence, throughout the proof we will always consider only arcs C such that $L_\rho(C) \leq \|\rho\| + 2\theta(\rho)(1)$.

It is known [6] that $u(\rho)$ is a convex metric on X such that

$$u(\rho)|_{\mathrm{dom}\rho \times \mathrm{dom}\rho} = \rho$$

and $u(\rho)$ is continuous with respect to r. It remains to prove that u is a continuous operator. For $B \subset X$ and $\varepsilon > 0$ let

$$S(B, \varepsilon) = \{y \in X : r(y, B) < \varepsilon\}.$$

If $C(x,y) \equiv C(y,x)$ is an r-rectifiable arc with endpoints $x, y \in X$ and $z \in C(x,y)$ we will use the notations $C(x,z)$ and $C(z,y)$ for the subarcs of $C(x,y)$ with endpoints $\{x,z\}$ and $\{y,z\}$ respectively.

Let $\rho \in \mathcal{CM}$ and let $\{\rho_n\}$ be a sequence in $\mathcal{CM}$ converging to ρ. Let $\mathrm{dom}\rho = B$, $\mathrm{dom}\rho_n = B_n$ for $n \in \mathbb{N}$ and let $\varepsilon > 0$.

Let $0 < \eta < \varepsilon/8$ be such that

$$\theta(\rho)(\eta) < \frac{\varepsilon}{16}. \tag{5}$$

We may also suppose that

$$|\rho(a,b) - \rho(x,y)| < \frac{\varepsilon}{32}$$

whenever $a, b, x, y \in B$ with $r(a,x) \leq 2\eta$ and $r(b,y) \leq 2\eta$ by uniform continuity of ρ.

Since Γ_{ρ_n} converges to Γ_ρ, θ is continuous and $\theta(\rho)(0) = 0$, the following four conditions hold for sufficiently large $n \in \mathbb{N}$:

(i) the Hausdorff distance between Γ_ρ and Γ_{ρ_n} is less than $\eta/4$;

(Note that this implies that the distance between B_n and B is less then $\eta/4$ in $\exp X$).

(ii) $x, y \in \mathrm{dom}\rho$, $x', y' \in \mathrm{dom}\rho_n$ with $r(x,x') \leq \eta$ and $r(y,y') \leq \eta$ implies $|\rho(x,y) - \rho_n(x',y')| < \varepsilon/16$;

Condition (ii) follows from (i) for we have $(a, b, \rho(a,b)) \in \Gamma_\rho$ with

$$r'[(a,b,\rho(a,b)),(x',y',\rho_n(x',y'))] < \frac{\eta}{4}.$$

Then $r(a,x) < 2\eta$ and $r(b,y) < 2\eta$ by the triangle inequality and this gives $|\rho(a,b) - \rho(x,y)| < \varepsilon/32$. Also, $|\rho(a,b) - \rho_n(x',y')| < \eta/4 < \varepsilon/32$ and so $|\rho(x,y) - \rho_n(x',y')| < \varepsilon/16$.

(iii) for an r-rectifiable arc C of r-length less than or equal to $2\theta(\rho)(1)$ with $C \subset X \setminus S(B, \eta/3)$ we have $|L_\rho(C) - L_{\rho_n}(C)| < \varepsilon/16$;

(iv) $\theta(\rho_n)(\eta) < \varepsilon/16$.

In the next two lemmas we prove that for $\gamma \in \mathcal{CM}$ in estimating $u(\gamma)$ it suffices to consider only arcs which close to the set domγ are “perpendicular” to it. I.e. $C(y,b) \in \mathcal{A}_\gamma(y,b)$ with $C(y,b) \cap B = \{b\}$ and $z \in C(y,b)$ close to b then $r(z,B)$ is equal to the r-length of $C(z,b)$.

Lemma 1. *Let $\gamma \in \mathcal{CM}$, $x \in \mathrm{dom}\gamma$ and $y \notin \mathrm{dom}\gamma$. Suppose that $\delta > 0$ with $\theta_\gamma(\delta) < \varepsilon/16$. Let $b \in \mathrm{dom}\gamma$ and $C(y,b) \in \mathcal{A}_\gamma(y,b)$. Then if $z \in C(y,b)$ with $r(z,\mathrm{dom}\gamma) < \delta$, $b' \in \mathrm{dom}\gamma$ such that $r(z,b') = r(z,\mathrm{dom}\gamma)$ and $[zb']$ is an r-segment we have*

$$L_\gamma(C(y,b)) + \gamma(b,x) + \frac{\varepsilon}{8} > L_\gamma(C(y,z)) + L_\gamma([zb']) + \gamma(b',x).$$

Proof. Using (4) we obtain

$$\begin{gathered} L_\gamma(C(y,z)) + L_\gamma([zb']) + \gamma(b',x) < \\ L_\gamma(C(y,z)) + \frac{\varepsilon}{16} + \gamma(b,x) + \gamma(b,b') \leq \\ L_\gamma(C(y,z)) + \gamma(b,x) + \sigma_\gamma(b,b') + \frac{\varepsilon}{16} \leq \\ L_\gamma(C(y,z)) + \gamma(b,x) + L_\gamma([zb'] \cup C(z,b)) + \frac{\varepsilon}{16} < \\ L_\gamma(C(y,z) \cup C(z,b)) + \gamma(b,x) + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = L_\gamma(C(y,b)) + \gamma(b,x) + \frac{\varepsilon}{8}. \end{gathered}$$

□

Lemma 2. *Let $\gamma \in \mathcal{CM}$, $x,y \in X \setminus \mathrm{dom}\gamma$. Suppose $a,b \in A$, $C(x,a) \in \mathcal{A}_\gamma(x,a)$ and $C(y,b) \in \mathcal{A}_\gamma(y,b)$. Let $\delta > 0$ with $\theta_\gamma(\delta) < \varepsilon/16$. If $z_1 \in C(x,a)$, $z_2 \in C(y,b)$ with $r(z_1,\mathrm{dom}\gamma) < \delta$, $r(z_2,\mathrm{dom}\gamma) < \delta$, $a',b' \in \mathrm{dom}\gamma$ such that $r(z_1,a') = r(z_1,\mathrm{dom}\gamma)$, $r(z_2,b') = r(z_2,\mathrm{dom}\gamma)$ and $[z_1a']$, $[z_2b']$ are r-segments then*

$$\begin{gathered} L_\gamma(C(x,z_1) \cup [z_1a']) + \gamma(a',b') + L_\gamma(C(y,z_2) \cup [z_2b']) < \\ L_\gamma(C(x,a)) + \gamma(a,b) + L_\gamma(C(y,b)) + \frac{\varepsilon}{4}. \end{gathered}$$

Proof. By Lemma 1

$$L_\gamma(C(y,b)) + \gamma(a,b) + \frac{\varepsilon}{8} > L_\gamma([z_2b'] \cup C(y,z_2)) + \gamma(a,b').$$

Again, by Lemma 1 we get

$$L_\gamma(C(x,a)) + \frac{\varepsilon}{8} > L_\gamma(C(x,z_1)\cup[z_1a']) + \gamma(a',a).$$

Adding these inequalities we obtain

$$L_\gamma(C(x,a)) + \gamma(a,b) + L_\gamma(C(y,b)) + \frac{\varepsilon}{4} >$$
$$L_\gamma([z_2b']\cup C(y,z_2)) + \gamma(a,b') + L_\gamma(C(x,z_1)\cup[z_1a']) + \gamma(a',a) \geq$$
$$L_\gamma(C(x,z_1)\cup[z_1a']) + L_\gamma([z_2b']\cup C(y,z_2)) + \gamma(a',b').$$

□

From here on let $n\in\mathbb{N}$ be such that conditions (i)–(iv) are satisfied.

Fix an arbitrary point $(x,y)\in X\times X$ such that $x\neq y$. We are going to show that

$$|u(\rho)(x,y) - u(\rho_n)(x,y)| < \varepsilon.$$

To simplify notations let $\alpha_\rho=\alpha, \beta_\rho=\beta$, $\alpha_{\rho_n}=\alpha_n$, $\beta_{\rho_n}=\beta_n$, $\sigma_\rho=\sigma$ and $\sigma_{\rho_n}=\sigma_n$. We need to consider several cases.

Case 1. $x,y\in B$. Then $u(\rho)(x,y)=\rho(x,y)$. Let $x_n,y_n\in B_n$ be such that $r(x,x_n)=r(x,B_n)$ and $r(y,y_n)=r(y,B_n)$. Then using (4), (ii), (iv) and the triangle inequality for $u(\rho_n)$ we have

$$u(\rho_n)(x,y) \leq L_{\rho_n}([xx_n]) + \rho_n(x_n,y_n) + L_{\rho_n}([y_ny]) <$$
$$\frac{\varepsilon}{16} + \Big(\rho(x,y) + \frac{\varepsilon}{16}\Big) + \frac{\varepsilon}{16} < u(\rho)(x,y) + \varepsilon.$$

Now,

$$u(\rho)(x,y) = \rho(x,y) < \rho_n(x_n,y_n) + \frac{\varepsilon}{16} \leq$$
$$u(\rho_n)(x,y) + L_{\rho_n}([xx_n]) + L_{\rho_n}([yy_n]) + \frac{\varepsilon}{16} < u(\rho_n)(x,y) + \varepsilon.$$

Case 2. $x\in B$ and $y\notin B$. Then $u(\rho)(x,y)=\alpha(x,y)$. We prove first that $u(\rho_n)(x,y) < u(\rho)(x,y)+\varepsilon$. Let $b\in B$ and $C(y,b)\in\mathcal{A}_\rho(y,b)$ be such that

$$L_\rho(C(y,b)) + \rho(x,b) < u(\rho)(x,y) + \frac{\varepsilon}{16}.$$

Let $C(y,b)$ have its natural linear order with initial point y. Let c' be the first point in $C(y,b)\cap\overline{S(B,\eta/3)}$. Take points $c\in B$, $c_n\in B_n$ such that

$r(c',c) = r(c',B)$ and $r(c',c_n) = r(c',B_n)$. Then $r(c,c_n) < \eta$. Let $x_n \in B_n$ be such that $r(x,x_n) = r(x,B_n)$. By Lemma 1,

$$\rho(x,c) + L_\rho([cc']) + L_\rho(C(y,c')) < \rho(x,b) + L_\rho(C(y,b)) + \frac{\varepsilon}{8} < u(\rho)(x,y) + \frac{\varepsilon}{4}. \qquad (*)$$

Now using (4), (ii), (iii) and (iv) we obtain

$$u(\rho_n)(x,y) \le L_{\rho_n}([xx_n]) + \rho_n(x_n,c_n) + L_{\rho_n}([c_nc']) + L_{\rho_n}(C(y,c')) < \frac{\varepsilon}{16} + \Big(\rho(x,c) + \frac{\varepsilon}{16}\Big) + \frac{\varepsilon}{16} + \Big(L_\rho(C(y,c')) + \frac{\varepsilon}{16}\Big) < u(\rho)(x,y) + \frac{\varepsilon}{2}.$$

The last inequality is true by $(*)$.

Now we prove that $u(\rho)(x,y) < u(\rho_n)(x,y) + \varepsilon$. We may suppose that $x,y \notin B_n$ or we are back in the first part of Case 2 or in Case 1 with the roles of ρ and ρ_n interchanged.

Suppose first that $u(\rho_n)(x,y) = \sigma_n(x,y)$. Let $C(x,y) \in \mathcal{A}_{\rho_n}(x,y)$ such that

$$L_{\rho_n}(C(x,y)) < \sigma_n(x,y) + \frac{\varepsilon}{8}. \qquad (**)$$

Let $C(x,y)$ have its natural linear order with initial point y. Let c' be the first point in $C(x,y) \cap \overline{S(B,\eta/3)}$. Take points $c \in B$, $c_n, x_n \in B_n$ such that $r(c',c) = r(c',B)$, $r(c',c_n) = r(c',B_n)$ and $r(x,x_n) = r(x,B_n)$. Then $r(c,c_n) < \eta$. Using conditions (4), (5), (ii), (iii) and (iv) we obtain

$$\begin{aligned} u(\rho)(x,y) = \alpha(x,y) &\le \rho(x,c) + L_\rho([cc']) + L_\rho(C(y,c')) < \\ &\Big(\rho_n(x_n,c_n) + \frac{\varepsilon}{16}\Big) + \frac{\varepsilon}{16} + \Big(L_{\rho_n}(C(y,c')) + \frac{\varepsilon}{16}\Big) \le \\ &\sigma_n(x_n,c_n) + L_{\rho_n}(C(y,c')) + \frac{3\varepsilon}{16} \le \\ &(L_{\rho_n}([xx_n]) + L_{\rho_n}(C(x,c')) + L_{\rho_n}([c'c_n])) + L_{\rho_n}(C(y,c')) + \frac{3\varepsilon}{16} < \\ &\frac{\varepsilon}{16} + L_{\rho_n}(C(x,y)) + \frac{\varepsilon}{16} + \frac{3\varepsilon}{16} < u(\rho_n)(x,y) + \frac{7\varepsilon}{16}. \end{aligned}$$

The last inequality is true by $(**)$.

Now let $u(\rho_n)(x,y) = \beta_n(x,y)$. There are points $a_n, b_n \in B_n$ and arcs $C(x,a_n) \in \mathcal{A}_{\rho_n}(x,a_n)$, $C(y,b_n) \in \mathcal{A}_{\rho_n}(y,b_n)$ such that

$$\beta_n(x,y) + \frac{\varepsilon}{8} > L_{\rho_n}(C(x,a_n)) + \rho_n(a_n,b_n) + L_{\rho_n}(C(y,b_n)). \qquad (***)$$

Let $C(y, b_n)$ have its natural linear order with initial point y. Let $c' \in C(y, b_n) \cap \overline{S(B, \eta/3)}$ be the first such point in $C(y, b_n)$. Let $c_n \in B_n$ be such that $r(c', c_n) = r(c', B_n)$, $x_n \in B_n$ be such that $r(x, x_n) = r(x, B_n)$ and $c \in B$ be such that $r(c', c) = r(c', B)$. Then using (4) and (iv) we have

$$\rho_n(a_n, x_n) \leq L_{\rho_n}(C(a_n, x)) + L_{\rho_n}([xx_n]) < L_{\rho_n}(C(a_n, x)) + \frac{\varepsilon}{16}$$

and $r(c_n, c) < \eta$.

By Lemma 2 applied to ρ_n and using (4), (5), (ii) and (iii) we obtain

$$u(\rho)(x, y) \leq L_\rho(C(y, c')) + L_\rho([cc']) + \rho(c, x) <$$
$$\left(L_{\rho_n}(C(y, c')) + \frac{\varepsilon}{16}\right) + \frac{\varepsilon}{16} + \left(\rho_n(c_n, x_n) + \frac{\varepsilon}{16}\right) + L_{\rho_n}([c'c_n]) + L_{\rho_n}([xx_n]) <$$
$$L_{\rho_n}(C(x, a_n)) + \rho_n(a_n, b_n) + L_{\rho_n}(C(y, b_n)) + \frac{\varepsilon}{8} + \frac{3\varepsilon}{16} < u(\rho_n)(x, y) + \frac{7\varepsilon}{16}.$$

The last inequality is true by $(***)$.

Case 3. $x, y \notin B \cup B_n$. Then

$$u(\rho)(x, y) = \min\{\sigma(x, y), \beta(x, y)\}$$

and

$$u(\rho_n)(x, y) = \min\{\sigma_n(x, y), \beta_n(x, y)\}.$$

Suppose first that $u(\rho)(x, y) = \beta(x, y)$.

Let $a, b \in B$ and $C(x, a) \in \mathcal{A}_\rho(x, a)$ and $C(y, b) \in \mathcal{A}_\rho(y, b)$ be such that

$$\beta(x, y) + \frac{\varepsilon}{2} > L_\rho(C(x, a)) + \rho(a, b) + L_\rho(C(y, b)).$$

Let $C(x, a)$ and $C(y, b)$ have natural linear orders with initial points x and y respectively. Let $e' \in C(x, a)$ be the first point with $e' \in \overline{S(B, \eta/3)}$ and let $c' \in C(y, b)$ be the first point with $c' \in \overline{S(B, \eta/3)}$. Let $e, c \in B$ and $e_n, c_n \in B_n$ be such that $r(e', B) = r(e', e)$, $r(c', B) = r(c', c)$, $r(e', B_n) = r(e', e_n)$, $r(c', B_n) = r(c', c_n)$. Then $r(e, e_n) < \eta$ and $r(c, c_n) < \eta$. By Lemma 2 we may suppose $a = e$ and $b = c$. Then

$$u(\rho_n)(x, y) \leq L_{\rho_n}(C(x, e') + L_{\rho_n}([e'e_n]) + \rho_n(e_n, c_n) + L_{\rho_n}([c_n c'])+$$
$$L_{\rho_n}(C(y, c')) < \left(L_\rho(C(x, e')) + \frac{\varepsilon}{16}\right) + \frac{\varepsilon}{16} + L_\rho([e'e]) + \left(\rho(e, c) + \frac{\varepsilon}{16}\right) +$$
$$L_\rho([c'c]) + \frac{\varepsilon}{16} + \left(L_\rho(C(y, c')) + \frac{\varepsilon}{16}\right) =$$
$$L_\rho(C(x, a)) + \rho(a, b) + L_\rho(C(y, b)) + \frac{5\varepsilon}{16} < u(\rho)(x, y) + \frac{13\varepsilon}{16}.$$

Now suppose that $u(\rho)(x,y) = \sigma(x,y)$. There is an arc $C(x,y) \in \mathcal{A}_\rho(x,y)$ such that

$$\sigma(x,y) + \frac{\varepsilon}{8} > L_\rho(C(x,y)).$$

If $C(x,y) \cap \overline{S(B,\eta/3)} = \varnothing$ then $C(x,y) \cap B_n = \varnothing$ and

$$u(\rho_n)(x,y) \le L_{\rho_n}(C(x,y)) < L_\rho(C(x,y)) + \frac{\varepsilon}{16} < \sigma(x,y) + \frac{3\varepsilon}{16}.$$

Now suppose that $C(x,y) \cap \overline{S(B,\eta/3)} \neq \varnothing$ and let $C(x,y)$ have its natural linear order with initial point x. Let $e', c' \in C(x,y)$ be the first and the last points from $\overline{S(B,\eta/3)}$, respectively. Let $e, c \in B$ and $e_n, c_n \in B_n$ be such that $r(e',B) = r(e',e)$, $r(c',B) = r(c',c)$, $r(e',B_n) = r(e',e_n)$, $r(c',B_n) = r(c',c_n)$. We obtain as before

$$\begin{gathered} u(\rho_n)(x,y) \le \\ L_{\rho_n}(C(x,e')) + L_{\rho_n}([e'e_n]) + \rho_n(e_n,c_n) + L_{\rho_n}([c'c_n]) + L_{\rho_n}(C(y,c')) < \\ \left(L_\rho(C(x,e')) + \frac{\varepsilon}{16}\right) + \frac{\varepsilon}{16} + \left(\rho(e,c) + \frac{\varepsilon}{16}\right) + \frac{\varepsilon}{16} + \left(L_\rho(C(y,c')) + \frac{\varepsilon}{16}\right) \le \\ L_\rho(C(x,e')) + \sigma_\rho(e,c) + L_\rho(C(y,c')) + \frac{5\varepsilon}{16} \le \\ L_\rho(C(x,e')) + (L_\rho([e'e]) + L_\rho(C(e',c')) + L_\rho([c'c])) + L_\rho(C(y,c')) + \frac{5\varepsilon}{16} < \\ L_\rho(C(x,y)) + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \frac{5\varepsilon}{16} < \sigma(x,y) + \frac{9\varepsilon}{16} < u(\rho)(x,y) + \varepsilon. \end{gathered}$$

The needed inequalities for all remaining cases can be shown similarly. In some cases the roles of ρ and ρ_n are interchanged. □

References

[1] T. Banakh, AE(0)–*spaces and regular operators extending (averaging) pseudometrics*, Bull. Polish Acad. Sci. Math. **42** (1994), no. 3, 197–206.

[2] T. Banakh , C. Bessaga, *On linear operators extending [pseudo]metrics*, Bull. Polish Acad. Sci. Math. **48** (2000), no. 1, 35–49.

[3] T. Banakh, N. Brodskiy, E.D. Tymchatyn, *On continuous extension of uniformly continuous functions and pseudometrics*, Preprint.

[4] T. Banakh, E.D. Tymchatyn, M. Zarichnyi, *Extensions of metrics: survey of results* (in preparation).

[5] C. Bessaga, *On linear operators and functors extending pseudometrics*, Fund. Math. **142** (1993), no. 2, 101-122.

[6] R.H. Bing, *A convex metric for a locally connected continuum*, Bull. Amer. Math. Soc. **55** (1949), 812–819.

[7] R.H. Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. **58** (1952), 536–556.

[8] E. Hewitt, K. Stromberg, *Real and Abstract Analysis*, Springer, Berlin, 1969.

[9] E. E. Moise, *Grille decomposition and convexification theorems for compact metric locally connected continua*, Bull. Amer. Math. Soc. **55** (1949), 1111–1121.

[10] E. E. Moise, *A Note of Correction*, Proc. Amer. Math. Soc. **2**, No.5 (1951), 838.

[11] I. Stasyuk, *Operators simultaneously extending partial ultrametrics*, Math. Methods and Phys.-Mech. Fields **49** (2006), no. 2, 27–32 (Ukraininan).

[12] E. N. Stepanova, *Continuation of continuous functions and the metrizability of paracompact p-spaces* (Russian), Mat. Zametki **53** (1993), no. 3, 92-101; translation in Math. Notes 53 (1993), no. 3-4, 308-314 MR 94k:54031.

[13] E.D. Tymchatyn, M. Zarichnyi, *On simultaneous linear extensions of partial (pseudo)metrics*, Proc. Amer. Math. Soc. **132** (2004), 2799–2807.

[14] E.D. Tymchatyn, M. Zarichnyi, *A note on operators extending partial ultrametrics*, Comment. Math. Univ. Carolinae **46** (2005), no. 3, 515–524.

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